The big slice phenomenon in Banach spaces

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Diameter 2 properties, Daugavet- and delta-points

Dissertation for the degree philosophiae doctor

University of Agder
Faculty of Engineering and Science
2021

Doctoral dissertations at the University of Agder: 335

ISSN: 1504-9272
ISBN: 978-82-8427-041-8

Printed by 07 Media
Kristiansand, Norway

Acknowledgments

First and foremost I want to express my deepest gratitude to my supervisors professors Trond Arnold Abrahamsen, Vegard Lima and Olav Kristian Gunnarson Dovland, for your invaluable guidance and continuous support. I also want to thank Peder Kristian Knutson, for convincing me to further pursue mathematics. Most likely, I would never have become a mathematician without your help and encouragement. Last but not least I want to thank my family, friends and colleagues for their support during the period of writing this thesis. My fiancé, Kristin, deserves special thanks for her patience and motivating words.

Kristiansand, Norway
April 2021
André Martiny

Publications

1. Trond A. Abrahamsen, Aleksander Leraand, André Martiny, and Olav Nygaard, *Two properties of Müntz spaces*, Demonstr. Math. **50** (2017), no. 1, 239–244.

2. André Martiny, *On Octahedrality and Müntz spaces*, Mathematica Scandinavica **126** (2020), no. 3, 513–518.

3. Trond A. Abrahamsen, Vegard Lima, André Martiny, and Stanimir Troyanski, *Daugavet- and delta-points in Banach spaces with unconditional bases*, Trans. Amer. Math. Soc. (to appear) (2021).

4. Trond A. Abrahamsen, Vegard Lima, and André Martiny, *Delta-points in Banach spaces generated by adequate families*, submitted (2021) .

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1 Introduction

1.1 Background

It is quite remarkable that there exist Banach spaces where *every* slice of the unit ball has a diameter of 2. Properties associated with certain subsets of the unit ball (e.g. slices, non-empty relatively weakly open subsets and finite convex combination of slices) having diameter 2, is what is meant by *the big slice phenomenon*. Banach spaces with diameter 2 properties exist in the "opposite world" of Banach spaces with the *Radon-Nikodým property*. Indeed, it is known that a Banach space X has the Radon-Nikodým property if and only if every closed and bounded convex subset of X has slices of arbitrarily small diameter. Reflexive, and in particular finite dimensional, Banach spaces have the Radon-Nikodým property. Therefore, the big slice phenomenon is purely infinite dimensional.

The simplest example of a Banach space where we meet the big slice phenomenon is probably c_0. Consider a norm one element x in c_0. Let $(e_n)_{n\in\mathbb{N}}$ be the standard basis. If we let $y_n = (x + e_n)/\|x + e_n\|$ and $z_n = (x - e_n)/\|x - e_n\|$, then we have norm one elements that converge weakly to x and whose distance tends to 2. In particular, slices and non-empty relatively weakly open subsets in the unit ball of c_0 have diameter 2. The above argument can be easily extended to show that even finite convex combinations of slices in the unit ball of c_0 have diameter 2. You just start with a finite number of norm one elements (instead of just x) and use the same e_n for each of these.

Possibly the first appearance of the big slice phenomenon in the literature came in connection with the so-called roughness of the norm. John and Zizler [JZ78] showed that if the norm of a Banach space X is 2-rough (equivalently locally octahedral [HLP15]), then the weak*-slices of B_{X^*} have diameter 2. Roughness of the norm was further investigated by Deville in 1988 in connection with octahedral norms. A Banach space has an octahedral norm if for any finite dimensional subspace E of X there exists a direction almost ℓ_1-orthogonal to E. Deville [Dev88] (see also [God89]) showed that if a Banach space X is octahedral, then the norm of X is 2-average rough, implying that finite convex combinations of weak*-slices of B_{X^*} have diameter 2.

The first paper devoted to investigating the diameter 2 properties was probably by Nygaard and Werner [NW01], where they showed that the non-empty relatively weakly open subsets of the unit ball in uniform algebras have diameter 2. Shvydkoy [Shv00] had previously shown that the same is true for such subsets of the unit ball of Banach spaces with the Daugavet property. Subsequent to [NW01], it was shown in [BGLPPRP04] that the non-empty relatively weakly open subsets of the unit ball of all M-embedded spaces have diameter 2, which generalizes the c_0 example mentioned above.

It can be argued that a common theory of diameter 2 properties began following [ALN13] where they did a systematic survey of previous research and introduced the following properties. A Banach space X has the *local diameter* 2 *property* (LD2P) if every slice of B_X has diameter 2, the *diameter* 2 *property* (D2P) if every non-empty relatively weakly open subset of B_X has diameter 2 and the *strong diameter* 2 *property* (SD2P) if every finite convex combination of slices of B_X has diameter 2. The SD2P implies the D2P by [GGMS87, Lemma II.1] and clearly the D2P implies the LD2P. The D2P was distinguished from the LD2P in [BGLPRZ15] and the SD2P and the D2P was shown, independently, to be different in [ABGLP15], [HL14] and [Oja14].

Other examples of Banach spaces where we can observe the big slice phenomenon are the Banach spaces c, ℓ_∞ and Banach spaces with the *Daugavet property* (e.g. $C[0,1]$, $L_1[0,1]$ and $L_\infty[0,1]$, see Section 1.2.3). All of the above examples enjoy the SD2P. It is worth pointing out that Banach spaces can enjoy diameter 2 properties in very different ways. Banach spaces with the symmetric strong diameter 2 property, satisfies that any finite number of slices have line segments of almost length 2 in a common direction. In a Banach space with the Daugavet property (and diametral local diameter 2 property), any norm one element in a slice lies as an endpoint of a line segment in the slice with length almost 2.

A Banach space has the Daugavet property (respectively diametral local diameter two property) if and only if for every norm one element x we have that every y in the unit ball (respectively x) is in the closed convex hull of the unit elements at a distance almost 2 from x. The Banach space $C[0,1]$ has the Daugavet property, and in particular $C[0,1]$ satisfies the SD2P. It is therefore natural to study what kind of diameter 2 structures certain subspaces of $C[0,1]$ inherit. In [ANP19] it was shown that extremely regular subspaces X of $C[0,1]$ have the Daugavet property and are *almost square*, i.e., for every finite dimensional subspace E of X, there is a direction almost ℓ_∞-orthogonal to E. In this thesis we single out and study the famous Müntz spaces (as subspaces of $C[0,1]$) and show that they have the strong diameter 2 property, but are neither locally almost square nor locally octahedral (it is pointed out in [HLLN19] that we actually show that Müntz spaces have the symmetric strong diameter 2 property). In addition, we show that Müntz spaces can be almost isometrically embedded into c and that Müntz spaces contain asymptotically isometric copies of c_0.

Pointwise versions of the Daugavet property and the diametral local diameter 2 property have recently been introduced and studied under the names of *Daugavet-* and *delta-points* respectively. Although the set of Daugavet- and delta-points coincide in $L_1(\mu)$, their preduals and for Müntz spaces [AHLP20], it is easy to construct examples of Banach spaces with delta-points that are not Daugavet-points. Indeed, in the space $C[0,1] \oplus_2 C[0,1]$, all points of the unit sphere are delta-points, however, no point is a Daugavet-point [AHLP20, Example 4.7]. In [Kad96, Corollary 2.3] Kadets proved that Banach spaces with the Daugavet property fail to have an unconditional basis. It also follows directly from the characterization in [IK04], that if a Banach space has the diametral local diameter 2 property and an unconditional basis, then the unconditional suppression basis constant must be at least 2. It is therefore natural to pose the question, do there exist Banach spaces with a 1-unconditional basis with delta-points, or even Daugavet-points? In order to study this problem, we investigate Banach spaces with 1-unconditional basis generated by so-called *adequate families*, denoted $h_{\mathcal{A},p}$. For $1 < p < \infty$, we show that neither $h_{\mathcal{A},p}$ nor $h^*_{\mathcal{A},p}$ contain any delta-points. In addition, we show that Banach spaces with a 1-subsymmetric basis can never have delta-points. The Banach spaces with a 1-subsymmetric basis is a big subclass of the Banach spaces with a 1-unconditional basis, and include well-known Banach spaces such as ℓ_p spaces, c_0, the subspaces h_M of Orlicz sequence spaces ℓ_m, the Schreier spaces and Lorentz sequence spaces $d(w,p)$ and their preduals $d(w,p)_*$.

More surprisingly, we show that there exist Banach spaces with a 1-unconditional basis with Daugavet-points. Moreover, we construct an $h_{\mathcal{A},1}$ space with "lots" of Daugavet-points in the sense that the Daugavet-points are weakly dense in the unit ball.

1.2 Summary of the thesis

In this section we summarize the main results of the thesis. Preliminary theory will be presented prior to each result. We begin, in Subsection 1.2.1, by discussing Müntz spaces, which is the focus of the two first papers, *"Two properties of Müntz spaces"* and *"Octahedrality and Müntz spaces"*. In Subsection 1.2.2, we then discuss diameter two properties which is the recurring theme throughout the thesis. We end the summary, with Subsection 1.2.3, by presenting the results related to Daugavet- and delta-points, which form the focus of the papers *"Daugavet- and delta-points in Banach spaces with unconditional bases"* and *"Delta-points in Banach spaces generated by adequate families"*. All the results are stated without proofs, but their origin is referenced where their proofs can be found in full detail.

The notation and terminology used throughout the thesis is standard (see e.g. [AK06]). If X is a Banach space, then B_X, S_X and X^* denote the unit ball, unit sphere and topological dual space, respectively. The convex hull of A of a subset of X is denoted $\operatorname{conv}(A)$ and the linear span by $\operatorname{span}(A)$. The norm- and weak-closure of A will be denoted $\overline{A}$ and $\overline{A}^w$, respectively.

1.2.1 Müntz spaces

The well-known *Weierstrass' approximation theorem* states that the polynomials are dense in $C[a,b]$, the space of continuous functions on the interval $[a,b]$, endowed with the sup-norm. In 1914, Herman Müntz generalized the Weierstrass approximation theorem by completely characterizing when a sequence of monomials are dense in $C[a,b]$. Let $\Lambda = (\lambda_i)_{i=0}^\infty$, with $\lambda_0 = 0$, be a strictly increasing sequence of non-negative real numbers and let $\Pi(\Lambda) := \operatorname{span}(t^{\lambda_i})_{i=0}^\infty \subseteq C[0,1]$.

Theorem 1.2.1 (cf. Müntz' Theorem [BE95, Theorem 4.2.1]). *Let* $\Lambda = (\lambda_i)_{i=0}^\infty$, *then* $\overline{\Pi(\Lambda)} = C[0,1]$ *if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

By Müntz' Theorem, we see that $M(\Lambda) := \overline{\Pi(\Lambda)}$ is a proper subspace of $C[0,1]$ whenever the series $\sum_{i=1}^\infty 1/\lambda_i$ converges. We will say that $\Lambda = (\lambda_i)_{i=0}^\infty$ is a *Müntz sequence* and $M(\Lambda)$ a *Müntz space* if $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Whenever the constants are excluded from $M(\Lambda)$, we denote the subspace as $M_0(\Lambda)$.

It is often too difficult to study Müntz spaces in general. Therefore Müntz spaces where Λ satisfies certain properties are frequently singled out. In particular, Müntz sequences being lacunary or satisfying the gap condition, i.e. $\inf \lambda_{k+1}/\lambda_k > 1$ or $\inf_{k\in\mathbb{N}}(\lambda_{k+1} - \lambda_k) > 0$, have been studied extensively (see e.g. [GL05]). It is known that a Müntz space $M(\Lambda)$ is isomorphic to a subspace of c_0 whenever Λ satisfies the gap condition (see Theorem 9.1.6(c) and Theorem 11.4.1 [GL05]). In [Wer00] Werner used the idea from the proof of Theorem 11.4.1 [GL05] (a proof due to Wojtaszczyk) to show that if Λ satisfies the gap condition, then $M(\Lambda)$ is almost isometric to a subspace of c (the space of convergent sequences), i.e., for each $\varepsilon > 0$ there exists an operator $J_\varepsilon : M(\Lambda) \to c$ such that

$$(1-\varepsilon)\left\|f\right\| \leq \left\|J_\varepsilon f\right\| \leq (1+\varepsilon)\left\|f\right\|.$$

The proof relies heavily on the fact that in this case every f in $M(\Lambda)$ is differentiable and that there exists for $0 < a < 1$ an upper bound $K(a)$ such that

$$\sup_{0\leq t\leq a} |f'(t)| \leq K(a)\left\|f\right\|.$$

The Bounded Bernstein inequality gives us a similar bound for the polynomials (which certainly are differentiable) when the Müntz sequence Λ satisfies $1 \leq \lambda_1$.

Theorem 1.2.2 (Bounded Bernstein inequality [BE97, Theorem 3.2])**.** *Let Λ be a Müntz sequence with $1 \leq \lambda_1$. Then for every $\varepsilon > 0$ there is a constant c_ε such that*

$$\sup_{0\leq t\leq 1-\varepsilon} |p'(t)| \leq c_\varepsilon \|p\|,$$

for all $p \in \Pi(\Lambda)$.

We use this upper bound to extend Werner's proof to get a general result about all Müntz spaces.

Theorem 3.2.6: [Mar]

Every Müntz space $M(\Lambda)$ can be written as a direct sum $X \oplus Y$ where X is finite dimensional and Y is almost isometric to a subspace of c.

In fact, as shown in Proposition 6.0.4 in the Appendix, it is possible to show that all Müntz spaces can be almost isometrically embedded into c.

Proposition 6.0.4

Let $M(\Lambda)$ be a Müntz space. Then for any $\varepsilon > 0$ there exists an operator $J_\varepsilon : M(\Lambda) \to c$ such that

$$(1-\varepsilon)\|f\| \leq \|J_\varepsilon f\| \leq \|f\|.$$

In [DLT96] the notion of an asymptotically isometric copy of c_0 was introduced.

Definition 1.2.3. cf. [DLT98] A Banach space X is said to contain an *asymptotically isometric copy of* c_0 if, for every null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$\sup_n (1-\varepsilon_n)|t_n| \leq \left\| \sum_{n\in\mathbb{N}} t_n x_n \right\| \leq \sup_n (1+\varepsilon_n)|t_n|,$$

for all finite sequences (t_n) of real numbers.

Naturally, by the definition, a Banach space containing an almost isometric copy of c_0 contains a copy of c_0. In [DLT98] they showed that ℓ_∞ can be equivalently renormed to fail to contain an asymptotically isometric copy of c_0. By James' distortion theorem [Jam64, Lemma 2.2] it is known that a Banach space X contains an almost isometric copy of c_0 as soon as it contains a copy of c_0. We can see that containing an asymptotically isometric copy of c_0 is a stronger property than containing an almost isometric copy of c_0. The study of Banach spaces containing asymptotically isometric copies of c_0 and ℓ_1 were initiated in [DLT96] and [DL97]. It was shown in [DL97] and [DLT96] that Banach spaces containing an asymptotically isometric copy of c_0 or ℓ_1 fail the fixed-point property. For Müntz spaces, we are able to say the following.

Theorem 2.2.6: [ALMN]

Müntz spaces contain asymptotically isometric copies of c_0.

1.2.2 Diameter two properties

Let X be a Banach space. A *slice* $S(x^*,\delta)$ of the unit ball B_X is a set

$$S(x^*,\delta) = \{y \in B_X \ : \ x^*(y) > \|x^*\| - \delta\},$$

where $x^* \in X^*$ and $\delta > 0$. If $(A_i)_{i=1}^N$ is a collection of subsets of X, then a convex combination of $(A_i)_{i=1}^N$ is a set

$$\sum_{i=1}^N \lambda_i A_i := \left\{ y \in X : y = \sum_{i=1}^N \lambda_i y_i,\ y_i \in A_i \right\},$$

where $\lambda_i \geq 0$ for all $i = 1, \ldots, N$ and $\sum_{i=1}^N \lambda_i = 1$.

Although the diameter two properties have appeared in the literature for several years in relation to roughness of the norm [Dev88], the Daugavet property [Shv00], uniform algebras [NW01] and M-embedded spaces [BGLPPRP04], it can be argued that a common theory of diameter 2 properties began with [ALN13]. In [ALN13] they introduced the following properties:

Definition 1.2.4. A Banach space X is said to have the

(i) *local diameter* 2 *property* (LD2P) if every slice of B_X has diameter 2;

(ii) *diameter* 2 *property* (D2P) if non-empty relatively weakly open subset of B_X has diameter 2;

(iii) *strong diameter* 2 *property* (SD2P) if every finite convex combination of slices has diameter 2.

Since every non-empty relatively weakly open subset of B_X contains a finite convex combination of slices [GGMS87, Lemma II.1], the SD2P implies the D2P. As every slice is a non-empty relatively weakly open subset of B_X, the D2P implies the LD2P as well. Note that none of the reverse implications hold. Indeed, $c_0 \oplus_2 c_0$ has the D2P by [ALN13, Theorem 3.2], but fail the SD2P by [ABGLP15, HL14, Oja14]. Thus the SD2P is a strictly stronger property than the D2P. The D2P is also strictly stronger than the LD2P, which can be seen by [BGLPRZ15, Theorem 2.4] or in Section 4.4 p. 69.

In *"Two properties of Müntz spaces"* we prove that all Müntz spaces have the SD2P, but our result is phrased in terms of octahedral norms.

Definition 1.2.5 (cf. [HLP15])**.** A Banach space X is said to have an *octahedral norm* if for every finite dimensional subspace F of X and every $\varepsilon > 0$, there exists $y \in S_X$ with

$$\|x + y\| \geq (1 - \varepsilon)(\|x\| + \|y\|)$$

for all $x \in F$.

The concept of an octahedral norm was introduced by Godefroy and Maurey (see [Dev88, p. 118]) where they showed that a Banach space X contains an isomorphic copy of ℓ_1 if and only if X admits an equivalent octahedral norm. Note that if a Banach space X has an octahedral norm, we will refer to X as octahedral when there is no risk of confusion. For a dual space X^*, a weak* slice is a slice of the form $S(x^{**}, \delta)$, where $x^{**} \in X^{**}$ is the canonical image of some $x \in X$.

Definition 1.2.6. Let X be a Banach space. We say that X^* has the

(i) $weak^*$ *local diameter* 2 *property* (w^*-LD2P) if every weak* slice of B_{X^*} has diameter 2;

(ii) $weak^*$ *diameter* 2 *property* (w^*-D2P) if every non-empty relatively weak* open subset of B_{X^*} has diameter 2;

(iii) $weak^*$ *strong diameter* 2 *property* (w^*-SD2P) if every convex combination of weak* slices of B_{X^*} has diameter 2.

Octahedral Banach spaces are intimately connected to the SD2P. The connection was already known in 1988 when Deville proved that if a Banach space X is octahedral, then X^* has the w^*-SD2P. However, the reverse implication was only stated without proof in [God89]. Therefore, proofs of the duality between octahedral Banach spaces and the SD2P appeared later.

Theorem 1.2.7 (cf. [BGLPRZ14] and [HLP15])**.** *Let X be a Banach space. Then:*

(i) X is octahedral if and only if X^* has the w^*-SD2P;

(ii) X has the SD2P if and only if X^* is octahedral.

Note that (ii) is an immediate consequence of (i) and the following result.

Proposition 1.2.8 (cf. [HLP15, Proposition 1.3])**.** *A Banach space X has the LD2P (respectively D2P, SD2P) if and only if X^{**} has the weak* LD2P (respectively weak* D2P, weak* SD2P).*

We now see that showing that the dual of any Müntz space is octahedral is equivalent to showing that any Müntz space has the SD2P.

Theorem 2.2.5: [ALMN]

The dual of any Müntz space is octahedral.

The authors of [HLP15] introduced two new weaker versions of octahedrality, in order to find a (pre)dual characterization of the w^*-D2P and the w^*-LD2P.

Definition 1.2.9. A Banach space X is

(i) *locally octahedral* if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that
$$\|x \pm y\| \geq 2 - \varepsilon;$$

(ii) *weakly octahedral* if for every finite-dimensional subspace E of X, every $x^* \in B_{X^*}$ and every $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\|x + y\| \geq (1 - \varepsilon)\left(|x^*(x)| + \|y\|\right) \qquad \text{for all } x \in E.$$

Note that octahedral Banach spaces are weakly octahedral. Furthermore, weakly octahedral Banach spaces are locally octahedral. None of the reverse implications hold because of the following relationships.

Theorem 1.2.10 (cf. [HLP15])**.** *Let X be a Banach space. Then*

(i) X is weakly octahedral if and only if X^* has the w^*-D2P;

(ii) X has the D2P if and only if X^* is weakly octahedral;

(iii) X is locally octahedral if and only if X^* has the w^*-LD2P;

(iv) X has the LD2P if and only if X^* is locally octahedral.

As any Müntz space can be embedded into c by Proposition 6.0.4, we see that the dual of any Müntz space is separable and thus has the RNP. Since this implies that the dual space of any Müntz space fail to have the w^*-LD2P we get the following result.

Theorem 3.3.1: [Mar]

No Müntz space $M(\Lambda)$ has a locally octahedral norm.

In [Kub14, Corollary 3.4] Kubiak showed that the Cesàro function spaces have the D2P. In addition, Kubiak showed that if X is a Banach space such that for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ with $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$, then X has the LD2P [Kub14, Proposition 2.5]. Furthermore, if in addition $y_n \to 0$ weakly, then X has the D2P [Kub14, Proposition 2.6]. This observation was the starting point of [ALL16] were they introduced the following properties:

Definition 1.2.11. Let X be a Banach space. Then X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$;

(ii) *weakly almost square* (WASQ) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly;

(iii) *almost square* (ASQ) if for every finite subset $(x_i)_{i=1}^N \subset S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x_i \pm y_n\| \to 1$ for every $i = 1, 2, \ldots, N$ and $\|y_n\| \to 1$.

Clearly WASQ implies LASQ and, in fact, ASQ implies WASQ, as the sequence in (iii) can be chosen to be weakly null (see [ALL16, Theorem 2.8]). Kubiak showed that if a Banach space X is WASQ (respectively LASQ), then X has the D2P (respectively LD2P). Proposition 2.5 in [ALL16] shows that ASQ also implies the SD2P. We can see that $C[0,1]$ fails to be LASQ. Indeed, by considering the constant function 1 it is clear that any $f \in S_{C[0,1]}$ satisfies $\max \|1 \pm f\| = 2$. This argument clearly also holds for any Müntz space containing the constant functions. To see that the result also holds for Müntz spaces $M_0(\Lambda)$ with $\lambda_1 < 1$ requires more effort, but can be seen by exploiting the Bounded Bernstein inequality, combined with an upper bound on the coefficients of the elements $\sum_{i=1}^k a_i t^{\lambda_i} \in \Pi(\Lambda)$.

Theorem 6.0.5: Appendix

Let X be $M(\Lambda)$ or $M_0(\Lambda)$ for any Müntz sequence Λ. Then X is not locally almost square.

Although there has been done a lot of study in the direction of ASQ Banach spaces (e.g. [ALL16, AHT20, BGLPRZ16, LLRZ17]), one question from [ALL16] still remains open: Is WASQ strictly stronger than LASQ?

1.2.3 Daugavet- and delta-points

In 1963 Daugavet proved that the equation

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{1.1}$$

holds for all compact operators T on $C[0,1]$ [Dau63], where Id is the identity operator on $C[0,1]$. Due to this discovery the equation (1.1) is now known as the *Daugavet equation*. Shortly after Daugavet established this equation for C[0,1] Lozanovskii [Loz66] proved that the Daugavet equation holds for all compact operators on $L_1[0,1]$. Banach spaces satisfying the Daugavet equation for all rank-1 operators $T : X \to X$ are said to have the *Daugavet property*. Subsequently, the Daugavet property has been further studied and characterized geometrically.

Lemma 1.2.12 ([KSSW00, Lemma 2.2])**.** *Let X be a Banach space. Then the following are equivalent:*

(i) X has the Daugavet property;

(ii) For every slice $S(x^*, \delta)$ of B_X, every $x \in S_X$ and every $\varepsilon > 0$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.

It is clear from (ii) that Banach spaces with the Daugavet property has the LD2P. In fact, the Daugavet property implies the SD2P, as shown in [ALN13, Theorem 4.4]. It is also known (see e.g. [BGLPRZ14, Lemma 2.3]) that the dual of a Banach space with the Daugavet property has the weak* SD2P. Typical examples of Banach spaces with the Daugavet property include $C[0,1]$, $L_1[0,1]$ and $L_\infty[0,1]$.

Note that there is a natural weakening of Lemma 1.2.12 (ii), by only considering slices containing x. This weakening is known as the *diametral local diameter* 2 *property*. The diametral local diameter 2 property was introduced in [IK04] under the name *space with bad projections* (see also [IK04] for unnamed appearances of this), but was also studied under the name local diameter 2 property+ in [AHN$^+$16]. The name diametral local diameter 2 property was first used in [BGLPRZ18].

Definition 1.2.13. Let X be Banach space. Then X has the *diametral local diameter* 2 *property* (DLD2P) if for every $x \in S_X$, every slice $S(x^*, \delta)$ with $x \in S(x^*, \delta)$ and $\varepsilon > 0$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.

The goal of the papers *"Daugavet- and delta-points in Banach spaces with unconditional bases"* and *"Delta-points in Banach spaces generated by adequate families"* is to study pointwise versions of the Daugavet property and the DLD2P:

Definition 1.2.14. Let X be a Banach space and let $x \in S_X$. We will say that

(i) x is a *delta-point* if for every slice $S(x^*, \delta)$ of B_X with $x \in S(x^*, \delta)$ and for every $\varepsilon > 0$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$;

(ii) x is a *Daugavet-point* if for every slice $S(x^*, \delta)$ of B_X and for every $\varepsilon > 0$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.

Daugavet- and delta-points have recently been introduced in [AHLP20] and studied further by several authors (e.g. [ALMT21, ALM20, DJR21, HPV21, JR20, MR20]). While the set of Daugavet-points and delta-points are not equal in general, they do coincide in $L_1(\mu)$, in spaces whose dual is isometric to $L_1(\mu)$ and in Müntz spaces (see [MR20, Theorem 3.2], [AHLP20, Theorem 3.1] and Theorem 6.0.7 in the Appendix). In this thesis we study Daugavet- and delta-points in Banach spaces with an unconditional basis.

Let X be a Banach space. Recall that a sequence $(e_i)_{i\in\mathbb{N}} \subset X$ is a *Schauder basis* for X if for every $x \in X$ there is a unique sequence $(x_i)_{i\in\mathbb{N}}$ of scalars, such that $x = \sum_{i\in\mathbb{N}} x_i e_i$. If X is a Banach space with a Schauder basis $(e_i)_{i\in\mathbb{N}}$, we say that $(e_i)_{i\in\mathbb{N}}$ is an *unconditional basis* if for every $x \in X$ its expansion $x = \sum_{i\in\mathbb{N}} x_i e_i$ converges unconditionally. Moreover, we say that $(e_i)_{i\in\mathbb{N}}$ is a *1-unconditional* basis if for all $N \in \mathbb{N}$ and all scalars $a_1, \ldots, a_N$, $b_1, \ldots, b_N$ such that $|a_i| \leq |b_i|$ for $i = 1, \ldots, N$, the following inequality holds,

$$\left\| \sum_{i=1}^{N} a_i e_i \right\| \leq \left\| \sum_{i=1}^{N} b_i e_i \right\|.$$

A basis $(e_i)_{i\in\mathbb{N}}$ is *normalized* if $\|e_i\| = 1$ for all $i \in \mathbb{N}$. For $x \in X$ the *support* of x is defined by $\operatorname{supp}(x) = \{i \in \mathbb{N} : e_i^*(x) \neq 0\}$, where $(e_i^*)_{i\in\mathbb{N}}$ are the biorthogonal functionals associated with $(e_i)_{i\in\mathbb{N}}$.

If $(e_i)_{i\in\mathbb{N}}$ is a 1-unconditional basis then for any subset $A \subset \mathbb{N}$ the projection P_A defined by

$$P_A\Big(\sum_{i\in\mathbb{N}} x_i e_i\Big) = \sum_{i\in A} x_i e_i$$

satisfies $\|P_A\| \leq 1$. The *unconditional suppression constant* is the supremum $\sup_A \|P_A\|$ over all subsets A of $\mathbb{N}$. Note that whenever $(e_i)_{i\in\mathbb{N}}$ is a 1-unconditional basis, then the unconditional suppression constant is also 1.

Our motivation for studying Daugavet- and delta-points in Banach spaces with an unconditional basis is based on the following two facts:

(i) Banach spaces with the Daugavet property fail to have an unconditional basis [Kad96, Corollary 2.3].

(ii) If a Banach space has the DLD2P and an unconditional basis, then the unconditional suppression basis constant must be at least 2 [IK04].

Note that (ii) follows directly from [IK04, Theorem 1.4], where it is shown that a Banach space X has the DLD2P if and only if

$$\|\mathrm{Id} - P\| \geq 2,$$

for every rank-1 projection P.

A 1-unconditional basis, $(e_i)_{i\in\mathbb{N}}$ is called *subsymmetric*, or 1-*subsymmetric*, if $\|\sum_{i\in\mathbb{N}} \theta_i x_i e_{k_i}\| = \|\sum_{i\in\mathbb{N}} x_i e_i\|$ for any $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$, any sequence of signs $(\theta_i)_{i\in\mathbb{N}}$, and any infinite increasing sequence of naturals $(k_i)_{i\in\mathbb{N}}$. The family of Banach spaces with a subsymmetric basis includes several well-known Banach spaces, such as c_0, the ℓ_p-spaces for $1 \leq p < \infty$, the Schreier spaces, the subspaces h_M of Orlicz sequence spaces ℓ_M and Lorentz sequence spaces $d(w,p)$ and their preduals $d(w,p)_*$. We show that the Banach spaces with a subsymmetric basis fail to have delta-points.

Theorem 4.2.17: [ALMT]

If X has subsymmetric basis $(e_i)_{i\in\mathbb{N}}$, then X has no delta-points.

Theorem 4.2.17 is a result of our study of so-called *minimal norming subsets*:

Definition 1.2.15. For any Banach space X with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ and for $x \in X$, define *the minimal norming subsets of* x as

$$M(x) := \{A \subseteq \mathbb{N} : \|P_A x\| = \|x\|, \|P_A x - x_i e_i\| < \|x\|, \text{ for all } i \in A\},$$

and

$$M^\infty(x) := \{A \in M(x) : |A| = \infty\}.$$

Our initial assumption was that no Banach space with a 1-unconditional basis can have delta-points. The concept of a minimal norming subset was the basis for our breakthrough in the study of delta-points in Banach spaces with a 1-unconditional basis. The importance of this concept can partially be explained through the following two results. For $D \in M^\infty(x)$, where $D = (d_i)_{i=1}^\infty$ and $d_i < d_{i+1}$, we define $D(n) = (d_i)_{i=1}^n$.

Lemma 4.2.14: [ALM]

Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ and let $x \in S_X$. Assume that there exists a slice $S(x^*, \delta)$, an $n \in \mathbb{N}$ and some $\eta > 0$ such that

(i) $x \in S(x^*, \delta)$;

(ii) $y \in S(x^*, \delta)$ implies that

$$\{i : |y_i| > \eta|x_i|,\ \operatorname{sgn} y_i = \operatorname{sgn} x_i\} \cap D(n) \neq \emptyset$$

for all $D \in M^\infty(x)$.

Then x is not a delta-point.

For Banach spaces with a subsymmetric basis, it can be shown that there exists a common coordinate k such that $k \in A$ for all $A \in M^\infty(x)$. Theorem 4.2.17 is therefore just an application of Lemma 4.2.14, where $x^* = e_k^*$. Furthermore, Lemma 4.2.14 shows that, in some sense, only the structure of the infinite sets of $M(x)$ is important for the existence of delta-points. This is even more clear from the following proposition.

Proposition 5.2.3: [ALM]

Let X be a Banach space with 1-unconditional basis. If for $x \in S_X$ there exists $n \in \mathbb{N}$ such that for $s = |\bigcup_{D\in M^\infty(x)} \{D(n)\}|$

$$\|P_E x\| > 1 - \frac{1}{2s} \quad \text{for all} \quad E \in \bigcup_{D\in M^\infty(x)} \{D(n)\},$$

then x is not a delta-point.

In particular, if $|M^\infty(x)| < \infty$, then x is not a delta-point.

Proposition 5.2.3 shows that in order for a Banach space with a 1-unconditional basis to have delta-points, the set $\bigcup_{D\in M^\infty(x)} \{D(n)\}$ must in some sense grow rapidly in size as n increases. To construct Banach spaces with such properties, we found it natural to study Banach spaces generated by *adequate families*.

Recall that a family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is an *adequate* family if

(i) $\mathcal{A}$ contains the empty set and the singletons;

(ii) $\mathcal{A}$ is hereditary: If $A \in \mathcal{A}$ and $B \subseteq A$, then $B \in \mathcal{A}$;

(iii) $\mathcal{A}$ is compact with respect to the topology of pointwise convergence: Given $A \subset \mathbb{N}$, if every finite subset of A is in $\mathcal{A}$, then $A \in \mathcal{A}$.

If $\mathcal{A}$ is an adequate family and $1 \leq p < \infty$, we define the norm $\|\cdot\|$ on c_{00}, the space of finitely supported sequences, by $\|(x_i)_{i\in\mathbb{N}}\| = \sup_{A\in\mathcal{A}} \left(\sum_{i\in A} |x_i|^p\right)^{1/p}$. The completion of c_{00} under the norm $\|\cdot\|$ is denoted $h_{\mathcal{A},p}$. It can be verified that the standard unit vectors $(e_i)_{i\in\mathbb{N}}$ form a normalized 1-unconditional basic sequence in $h_{\mathcal{A},p}$.

If $\mathcal{A} = \mathcal{P}(\mathbb{N})$ we can see that $h_{\mathcal{A},1} = \ell_1$, and if $\mathcal{A} = \{\emptyset, \{n\} : n \in \mathbb{N}\}$, then $h_{\mathcal{A},1} = c_0$. The class of $h_{\mathcal{A},p}$ spaces includes the Schreier spaces, ℓ_p spaces for $1 \leq p < \infty$, c_0 and $\ell_1(c_0)$. If $\mathcal{A}$ is an adequate family of purely finite subsets of $\mathbb{N}$ or if $p > 1$, then $h_{\mathcal{A},p}$ spaces cannot contain delta-points. In fact, we can say more:

Theorem 5.3.1: [ALM]

Let $\mathcal{A}$ be an adequate family of subsets of $\mathbb{N}$ and let $1 < p < \infty$. Then

(i) $h_{\mathcal{A},p}$ does not have delta-points;

(ii) $h^*_{\mathcal{A},p}$ does not have delta-points.

An important step towards understanding Daugavet- and delta-points in spaces with a 1-unconditional basis came by studying an $h_{\mathcal{A},1}$ space generated by an adequate family defined in the following way. Begin by partitioning $\mathbb{N}$ into blocks of integers $\{1, 2\}$ and $\{3n, 3n + 1, 3n + 2\}$ for $n \in \mathbb{N}$. Then declare A to be in $\mathcal{A}$ if and only if A contains at most one element from each block and whenever A contains an integer k divisible by 3, then A does not contain any integer bigger than k. More formally, define $B_0 = \{1, 2\}$ and $B_n = \{3n, 3n + 1, 3n + 2\}$ for $n \in \mathbb{N}$. Let $\mathcal{A}$ be such that $A \in \mathcal{A}$ if A satisfies $|A\cap B_n| \leq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and if $A\cap\{3n\} = \{3n\}$, then $A \cap \{1, 2, \ldots, 3n\} = A$. Note that this adequate family can be visualized as a graph, as shown in Figure 1.1. An element of $\mathcal{A}$ can, with this picture, be realized as a subset of a path starting from either 1 or 2, where the multiples of 3 are "dead ends". For the sake of reference, let us call $\mathcal{A}$ the *block-three family*.

For example, by the definition of the norm on $h_{\mathcal{A},1}$, we can see that the element

$$x = (2^{-1}, 2^{-1}, 2^{-1}, 2^{-2}, 2^{-2}, 2^{-2}, \ldots, 2^{-n}, 2^{-n}, 2^{-n}, \ldots),$$

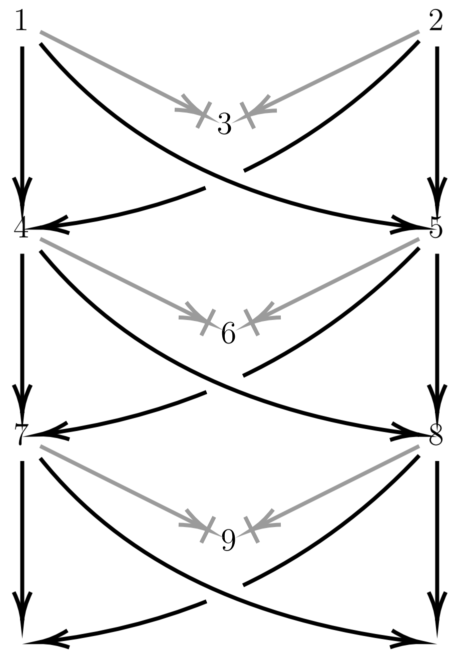

Figure 1.1: Visualizing the block-three family.

is of norm 1. Indeed, if $A \in \mathcal{A}$ is infinite, we can assume A is such that $|A \cap B_n| = 1$ for all $n \in N$, then $\sum_{i \in A} x_i = \sum_{i=1}^{\infty} 2^{-i} = 1$. If A is finite, we can also see that $\sum_{i \in A} x_i \leq 1$. Note that there are uncountably many ways of choosing a set A of infinite cardinality, such that $A \in M(x)$. In fact, x serves as an example of a vector where we cannot use Proposition 5.2.3 to conclude that x is not a delta-point. Although the block-three family seemed promising, it is possible to show that no element in $h_{\mathcal{A},1}$ is a delta-point. We omit the details, but the key observation from our argument was that the elements of the block-three family are too "intertwined" (see Figure 1.1). In particular, the fact that any natural number n can essentially be connected with any natural number bigger than n, seemed to prevent $h_{\mathcal{A},1}$ from containing delta-points.

To construct an adequate family similar to the block-three family, but with "less" overlap between the elements of the adequate family, we used the binary tree $\mathfrak{B}$ and constructed the $h_{\mathcal{A},1}$ space, $X_{\mathfrak{B}}$, which has some surprising properties. The adequate family constructed by the binary tree can be visualized in the same manner as the block-three family, see Figure 1.2.

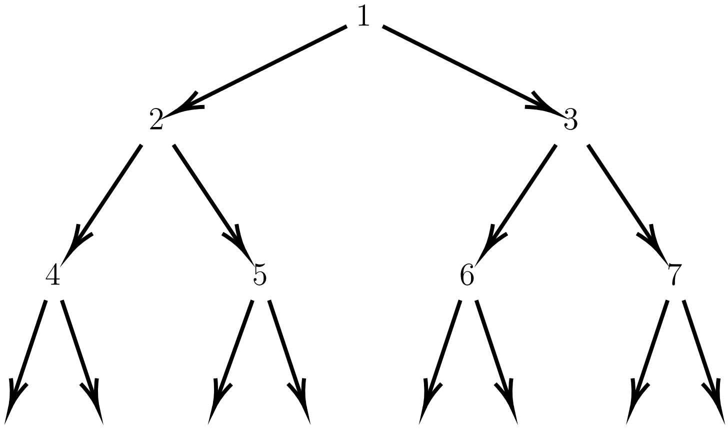

Figure 1.2: The binary tree.

Theorem 4.3.1: ALMT

In $X_{\mathfrak{B}}$ we have that

(i) $X_{\mathfrak{B}}$ has a delta-point;

(ii) $X_{\mathfrak{B}}$ does not have Daugavet-points.

We then modified the adequate family generated by the binary tree slightly, by removing the root and implementing more structure from the block-three family, to construct an even more interesting Banach space, *the modified binary tree space* $X_{\mathfrak{M}}$. The modified binary tree is slightly harder to visualize compared to the block-three family and the binary tree, as it includes paths with "dead ends", as shown in gray in Figure 1.3. An element in the modified binary tree can be seen as a subset of a path starting from either 1 or 2, and if you move along a gray path you reach a dead end, e.g., the path going from 1 to 3 to 4 is a maximal path, as it uses a gray path.

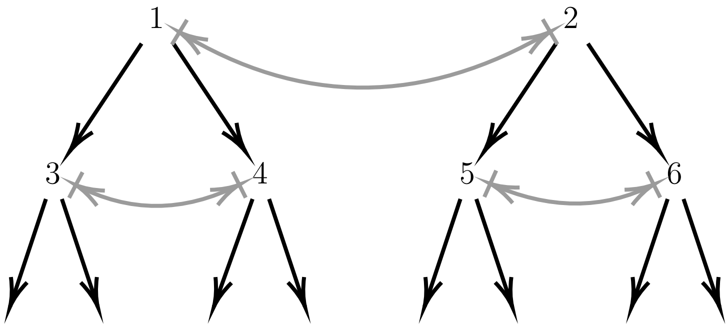

Figure 1.3: The modified binary tree.

Theorem 4.4.4: [ALMT]

In $X_{\mathfrak{M}}$ we have that

(i) there exists $x \in S_{X_{\mathfrak{M}}}$ which is a Daugavet-point;

(ii) there exists $w \in S_{X_{\mathfrak{M}}}$ which is a delta-point, but not a Daugavet-point.

In addition, the modified binary tree space has the LD2P, but fails to have the D2P and is neither LASQ nor locally octahedral.

Define

$$\mathfrak{F} = \{z \in S_{X_{\mathfrak{M}}} : z_i \in \{0, \pm 1\} \text{ for all } i \in \mathbb{N}\},$$

and

$$E_{X_{\mathfrak{M}}} = \left\{ E \subset \mathbb{N} : \sum_{i \in E} e_i \in S_X \right\}.$$

Note that the key difference between the binary tree and the modified binary tree (disregarding the deviation in the root) is that there are additional sets in the adequate family $\mathfrak{M}$. The implication of this, is that several norm one elements of $X_{\mathfrak{B}}$ has a norm greater than one, when viewed in $X_{\mathfrak{M}}$. For example, let $y = e_1 + e_2$ where e_1 and e_2 are the basis vectors corresponding to the two roots of the modified binary tree. Then y has norm 2 in $X_{\mathfrak{M}}$, but the corresponding element in $X_{\mathfrak{B}}$, $y = e_2 + e_3$, has norm one. We can see that y is in some sense maximal in $B_{X_{\mathfrak{B}}}$ as there does not exist $i \in \mathbb{N}$ such that $y \pm e_i \in B_{X_{\mathfrak{B}}}$. However, because there are more sets in the adequate family associated with $\mathfrak{M}$, no $z \in \mathfrak{F}$ is maximal as in the above sense.

Theorem 4.4.2: [ALMT]

Let $x \in S_{X_{\mathfrak{M}}}$, then the following are equivalent:

(i) x is a Daugavet-point;

(ii) $\|x - P_E x\| = 1$ for all $E \in E_{X_{\mathfrak{M}}}$;

(iii) for any $z \in \mathfrak{F}$, either $\|x - z\| = 2$ or for all $\varepsilon > 0$ there exists $s \in \mathfrak{M}$ such that $z \pm e_s \in \mathfrak{F}$ and $\|x - z \pm e_s\| > 2 - \varepsilon$.

The theorem above provides us with a tool to show that there are "lots" of Daugavet points in $X_{\mathfrak{M}}$ in the following sense:

Theorem 4.4.7: [ALMT]

In $X_{\mathfrak{M}}$ every non-empty relatively weakly open subset of $B_{X_{\mathfrak{M}}}$ contains a Daugavet-point.

We can see that this implies that the Daugavet-points in $S_{X_{\mathfrak{M}}}$ are weakly dense in $B_{X_{\mathfrak{M}}}$. As Banach spaces with the Daugavet property cannot have an unconditional basis, this leads to the following natural question:

Question 1. Given a Banach space X. How “massive” does the set of Daugavet-points in S_X have to be in order to ensure that X fails to have an unconditional basis?

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2 Two Properties of Müntz spaces

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Published in *Demonstratio Matematicae*, October 2017, volume 50, issue 1, pp. 239–244.

Abstract

We show that Müntz spaces, as subspaces of $C[0,1]$, contain asymptotically isometric copies of c_0 and that their dual spaces are octahedral.

2.1 Introduction

Let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of non-negative real numbers and let $M(\Lambda) = \overline{\text{span}}\{t^{\lambda_k}\}_{k=0}^{\infty} \subset C[0,1]$ where $C[0,1]$ is the space of real valued continuous functions on $[0,1]$ endowed with the max-norm. We will call $M(\Lambda)$ a Müntz space provided $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$. The name is justified by Müntz' wonderful discovery that if $\lambda_0 = 0$ then $M(\Lambda) = C[0,1]$ if and only if $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$.

It is well known that $C[0,1]$ contains isometric copies of c_0 (see e.g. [AK06, p. 86] how to construct them) and that its dual space is isometric to an $L_1(\mu)$ space for some measure μ. The aim of this paper is to demonstrate that Müntz spaces inherit quite a bit of structure from $C[0,1]$ in that they always contain asymptotically isometric copies of c_0, and that their dual spaces are always octahedral. (An $L_1(\mu)$ space is octahedral. See below for an argument.) Let us proceed by recalling the definitions of these two concepts and put them into some context.

Definition 2.1.1. [DLT98, Theorem 2] A Banach space X is said to contain an *asymptotically isometric copy of* c_0 if there exist a sequence $(x_n)_{n=1}^{\infty}$ in X and constants $0 < m < M < \infty$ such that for all sequences $(t_n)_{n=1}^{\infty}$ with finitely many non zero terms

$$m \sup_n |t_n| \leq \left\| \sum_n t_n x_n \right\| \leq M \sup_n |t_n|,$$

and

$$\lim_{n\to\infty} \|x_n\| = M.$$

R. C. James proved a long time ago (see [Jam64]) that X contains an almost isometric copy of c_0 as soon as it contains a copy of c_0. Note that containing an asymptotically isometric copy of c_0 is a stronger property, see e.g. [DLT98, Example 5].

Definition 2.1.2. A Banach space X is said to be *octahedral* if for any finite-dimensional subspace F of X and every $\varepsilon > 0$, there exists $y \in S_X$ with

$$\|x + y\| \geq (1-\varepsilon)(\|x\| + 1) \text{ for all } x \in F.$$

This concept was introduced by G. Godefroy and B. Maurey (see [Dev88, p. 118]), and in [God89] the following result can be found on page 12:

Theorem 2.1.3 (Deville-Godefroy)**.** *Let X be a Banach space. Then X^* is octahedral if and only if every finite convex combination of slices of B_X has diameter 2.*

By a slice of B_X we mean a set of the form

$$S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon, \varepsilon > 0, x^* \in S_{X^*}\}.$$

Remark 2.1.4. As we have mentioned, Theorem 2.1.3 can be found, but without proof, in [God89]. Deville had proven in [Dev88, Theorem 1 and Proposition 3] that if X is octahedral, every finite convex combination of w^*-slices of B_{X^*} has diameter 2. In the same paper he asks if the converse is true (Remark (c) on page 119). Since there is no proof included in [God89], new proofs appeared, independently, in [BLR14] and [HLP15], in connection with a new study of spaces where all finite convex combination of slices of B_X has diameter 2.

When we show that the dual of Müntz spaces are octahedral we will use Theorem 2.1.3 and establish the equivalent property stated there. Note that an $L_1(\mu)$ space is octahedral. Indeed, the bidual of such a space can be written $L_1(\mu)^{**} = L_1(\mu) \oplus_1 X$

for some subspace X of $L_1(\mu)^{**}$ (see e.g. [HWW93, IV. Example 1.1]). From here the octahedrality of $L_1(\mu)$ is a straightforward application of the Principle of Local Reflexivity.

The main reference concerning Müntz spaces is [GL05]. But there most of the phenomena that are studied are linked to spreading properties of Λ and not general results concerning all Müntz spaces.

We do not know of much research in the direction of our results. But we would like to mention a paper of P. Petráček ([Pet12]), where he demonstrates that Müntz spaces are never reflexive and asks whether they can have the Radon-Nikodým property. Since the Radon-Nikodým property implies the existence of slices of arbitrarily small diameter, we now understand that Müntz spaces rather belong to the "opposite world" of Banach spaces.

See also Remark 2.2.9 for some more related results.

2.2 Results

Definition 2.2.1. We will say that a strictly increasing sequence of non-negative real numbers $(\lambda_k)_{k=0}^{\infty}$ has the *Rapid Increase Property (RIP)* if $\lambda_{k+1} \geq 2\lambda_k$ for every $k \geq 0$.

We will call a function of the form

$$p(x) = x^{\alpha} - x^{\beta},$$

where $0 \leq \alpha < \beta$, a *spike function*.

Remark 2.2.2. If $\alpha > 0$ it should be clear that any spike function p satisfies $p(0) = p(1) = 0$, attains its norm on a unique point x_p, is strictly increasing on $[0, x_p]$, and strictly decreasing on $[x_p, 1]$. To visualize the arguments that come, we think it is a good idea at this stage to draw the graphs of e.g. $x^{100} - x^{200}$ and $x^{1000} - x^{20000}$.

We will need the following result below.

Lemma 2.2.3. *Let $(\lambda_k)_{k=0}^{\infty}$ be an RIP sequence and $(p_k)_{k=0}^{\infty}$ the sequence of corresponding spike functions $p_k(x) = x^{\lambda_k} - x^{\lambda_{k+1}}$. Then* $\inf_k \|p_k\| \geq 1/4$. *Moreover, the sequence $(p_k/\|p_k\|)_{k=1}^{\infty}$ converges to 0 weakly in $M(\Lambda)$.*

Proof. We want to find the norm of the spike function defined by

$$p_k(x) = x^{\lambda_k} - x^{\lambda_{k+1}}.$$

Observe that $r_k(x) := x^{\lambda_k} - x^{2\lambda_k} \leq p_k(x)$ for all $x \in [0,1]$. Now, by standard calculus, r_k attains its maximum at x_k where $x_k^{\lambda_k} = \frac{1}{2}$. Thus

$$\|p_k\| \geq r_k(x_k) = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}.$$

As $(p_k)_{k=1}^{\infty}$ converges pointwise to 0 and $\inf_k \|p_k\| \geq 1/4$, the sequence $(p_k/\|p_k\|)_{k=1}^{\infty}$ converges pointwise to 0 and thus weakly to 0 as it is bounded. □

Remark 2.2.4. By standard calculus one can show that the point at which p_k in Lemma 2.2.3 obtains its norm is $\bar{x}_k = (\lambda_k/\lambda_{k+1})^{1/(\lambda_{k+1}-\lambda_k)}$. For sufficiently large λ_k it is straightforward to show that

$$y_k := 1/(\lambda_{k+1} - \lambda_k)^{1/(\lambda_{k+1}-\lambda_k)} \leq \bar{x}_k,$$

that y_k is strictly monotone, and that y_k converges to 1 ($\lambda_k \geq 3$ is sufficient).

Theorem 2.2.5. *The dual of any Müntz space is octahedral.*

Proof. Let $M(\Lambda)$ be a Müntz space. Let

$$C = \sum_{j=1}^{n} \mu_j S(x_j^*, \varepsilon_j),$$

where $\sum_{j=1}^{n} \mu_j = 1, \mu_j > 0$, and $S(x_j^*, \varepsilon_j), 1 \leq j \leq n$, is a slice of $B_{M(\Lambda)}$. We will show that the diameter of C is 2 (cf. Theorem 2.1.3). To this end, start with some $f \in C$ and write $f = \sum_{j=1}^{n} \mu_j g^j$, where $g^j \in S(x_j^*, \varepsilon_j)$. Let $(\lambda_k)_{k=0}^{\infty}$ be an RIP subsequence of Λ (which is possible as $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$) and put

$$h_k^{j+} = g^j + (1 - g^j(x_k))\frac{p_k}{\|p_k\|}$$
$$h_k^{j-} = g^j - (1 + g^j(x_k))\frac{p_k}{\|p_k\|}$$

where $(p_k)_{k=0}^{\infty}$ is the sequence of spike functions corresponding to $(\lambda_k)_{k=0}^{\infty}$ and x_k the (unique) point where p_k attains its norm. We will prove that, for any $\varepsilon > 0$, there exists a $K = K(\varepsilon)$ such that whenever $k \geq K$ we have $\frac{1}{1+2\varepsilon}h_k^{j+}, \frac{1}{1+2\varepsilon}h_k^{j-} \in S(x_j^*, \varepsilon_j)$ for every $1 \leq j \leq n$. Then, clearly

$$\frac{1}{1+2\varepsilon} \sum_{j=1}^{n} \mu_j h_k^{j\pm} \in C,$$

and

$$\left\| \frac{1}{1+2\varepsilon} \sum_{j=1}^{n} \mu_j h_k^{j+} - \frac{1}{1+2\varepsilon} \sum_{j=1}^{n} \mu_j h_k^{j-} \right\|$$
$$\geq \frac{1}{1+2\varepsilon} \left(\sum_{j=1}^{n} \mu_j [h_k^{j+}(x_k) - h_k^{j-}(x_k)] \right) = \frac{2}{1+2\varepsilon}.$$

for all $k \geq K$. Since ε is arbitrary, we can thus conclude that C has diameter 2.

To produce the $K = K(\varepsilon)$ above, note that $h_k^{j\pm}$ converges to g^j pointwise, and thus weakly since the sequences are bounded. As $U_j := \{x \in M(\Lambda) : x_j^*(x) > 1 - 2\varepsilon_j\}$ is weakly open, each sequence $(h_k^{j\pm})_{k=0}^{\infty}$ enters U_j eventually. Since there are only a finite number of sets U_j, this entrance is uniform. So, what is left to prove is that for $\varepsilon > 0$ there exists K such that $\|h_k^{j\pm}\| \leq 1 + 2\varepsilon$ whenever $k \geq K$.

Now, let $\varepsilon > 0$. Combining Remark 2.2.2, Remark 2.2.4, that $(p_k/\|p_k\|)_{k=1}^{\infty}$ converges pointwise to 0, and the continuity of g^j, we can find $K \in \mathbb{N}$ such that for all $k \geq K$ there are points $0 < a_k < x_k < b_k < 1$ such that

$$\frac{p_k(x)}{\|p_k\|} > \varepsilon \Leftrightarrow x \in (a_k, b_k),$$
$$\sup_{u,v \in (a_k,b_k)} |g^j(u) - g^j(v)| < \varepsilon,\ j = 1, \ldots, n.$$

We will see that this K does the job for the given $\varepsilon > 0$: Let $k \geq K$ and suppose $x \notin (a_k, b_k)$. Then

$$|h_k^{j+}(x)| = \left| g^j(x) + (1 - g^j(x_k)) \frac{p_k(x)}{\|p_k\|} \right| \leq |g^j(x)| + 2\varepsilon \leq 1 + 2\varepsilon.$$

If $x \in (a_k, b_k)$, observe that

$$|h_k^{j+}(x)| \leq \left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| + |g^j(x) - g^j(x_k)| \frac{p_k(x)}{\|p_k\|}$$
$$< \left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| + \varepsilon.$$

Now, if $g^j(x) \geq 0$, then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq g^j(x) + (1 - g^j(x)) = 1.$$

If $g^j(x) < 0$ and $g^j(x) + (1 - g^j(x))p_k(x)/\|p_k\| \geq 0$, then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq g^j(x) + (1 - g^j(x)) = 1.$$

If $g^j(x) < 0$ and $g^j(x) + (1 - g^j(x))p_k(x)/\|p_k\| < 0$, then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq |g^j(x)| \leq 1.$$

In any case we have for $k \geq K$ and $x \in [0,1]$ that $|h_k^{j+}(x)| \leq 1 + 2\varepsilon$. The argument that $\|h_k^{j-}\| \leq 1 + 2\varepsilon$ is similar. □

Theorem 2.2.6. *Müntz spaces contain asymptotically isometric copies of* c_0.

Proof. We will construct a sequence $(f_n)_{n=1}^\infty \subset M(\Lambda)$ and pairwise disjoint intervals $I_n = (a_n, b_n) \subset [0,1]$ such that for all $n \in \mathbb{N}$

(i) $f_n(x) \geq 0$ for all $x \in [0,1]$,

(ii) $\|f_n\| = 1 - 1/2^n$,

(iii) $b_n < a_{n+1}$

(iv) $f_n(x) > 1/2^{2n} \Leftrightarrow x \in I_n$,

(v) $f_n(x) < 1/2^{2m}$ whenever $m \geq n$ and $x \in I_m$.

To this end choose a subsequence of Λ with the RIP. For simplicity denote also this subsequence by $(\lambda_k)_{k=0}^\infty$. Let $(p_k)_{k=1}^\infty$ be its corresponding sequence of spike functions, and let x_k be the (unique) point in $(0,1)$ where p_k obtains its maximum.

Now, start by letting $k_1 = 1$ and put

$$f_1 = (1 - 1/2)\frac{p_{k_1}}{\|p_{k_1}\|}.$$

Using continuity and properties of p_1, we can find an interval $I_1 = (a_1, b_1)$ such that $0 < a_1 < b_1 < 1$ and $f_1(x) > \frac{1}{2^2} \Leftrightarrow x \in I_1$. By construction f_1 satisfies the conditions (i) - (iv).

To construct f_2 we use Lemma 2.2.3 and Remarks 2.2.2 and 2.2.4 to find $k_2 \in \mathbb{N}$ and an interval $I_2 = (a_2, b_2)$ with $b_1 < a_2 < b_2 < 1$ such that

$$x \in I_2 \Leftrightarrow p_{k_2}(x) > \frac{1/2^4}{1 - 1/2^2}\|p_{k_2}\|,$$
$$x \in I_2 \Rightarrow p_{k_1}(x) \leq \frac{1}{2^4}.$$

Let

$$f_2 = (1 - 1/2^2)\frac{p_{k_2}}{\|p_{k_2}\|}.$$

By construction f_1 now satisfies condition (v) for $m \le 2$ and f_2 satisfies conditions (i) - (iv).

To construct f_3 we use Lemma 2.2.3 and Remarks 2.2.2 and 2.2.4 again to find $k_3 \in \mathbb{N}$ and an interval $I_3 = (a_3, b_3)$ with $b_2 < a_3 < b_3 < 1$ such that

$$x \in I_3 \Leftrightarrow p_{k_3}(x) > \frac{1/2^6}{1 - 1/2^3}\|p_{k_3}\|,$$
$$x \in I_3 \Rightarrow p_{k_j}(x) \le \frac{1}{2^6} \quad \text{for } j = 1, 2.$$

Let

$$f_3 = (1 - 1/2^3)\frac{p_{k_3}}{\|p_{k_3}\|}.$$

By construction f_1 and f_2 now satisfy condition (v) for $m \le 3$ and f_3 satisfies conditions (i) - (iv). If we continue in the same manner we obtain a sequence $(f_n)_{n=1}^{\infty} \subset M(\Lambda)$ and a sequence of intervals $I_n = (a_n, b_n)$ which satisfies the conditions (i) - (v).

Now we will show that $(f_n)_{n=1}^{\infty}$ satisfies the requirements of Definition 2.1.1. To this end we need to find constants $0 < m < M < \infty$ such that given any sequence $(t_n)_{n=1}^{\infty}$ with finitely many non zero terms

$$m \sup_n |t_n| \le \|\sum_n t_n f_n\| \le M \sup_n |t_n| \tag{2.1}$$

and

$$\lim_{n\to\infty} \|f_n\| = M \tag{2.2}$$

We claim that (2.1) and (2.2) holds with $m = \frac{1}{4}$ and $M = 1$. First observe that we have $\lim_{n\to\infty} \|f_n\| = 1$ immediately from the requirements, so (2.2) holds for $M = 1$. In order to prove the two inequalities in (2.1), let $(t_n)_{n=1}^{\infty}$ be an arbitrary sequence with finitely many non zero terms. First we will prove that $1/4 \sup_n |t_n| \le \left\|\sum_n t_n f_n\right\|$. We can assume by scaling that $\sup |t_n| = 1$. Since $(t_n)_{n=1}^{\infty}$ has finitely many non zero terms, its norm is attained at, say, $n = N$, i.e. $|t_N| = 1$. Put $x_N = x_{k_N}$ where x_{k_N} is the point where p_{k_N} and thus f_N attains its norm. Then

$$\begin{aligned}
\|\sum_{n\in\mathbb{N}} t_n f_n\| &\ge |t_N f_N(x_N)| - |\sum_{n\ne N} t_n f_n(x_N)| \\
&\ge 1 - \frac{1}{2^N} - \sum_{n\ne N} |f_n(x_N)| \\
&> 1 - \frac{1}{2^N} - \frac{1}{4} \ge \frac{1}{4}.
\end{aligned}$$

We conclude that the left hand side of the inequality (2.1) holds. Now we will show the right hand side of this inequality holds, i.e. we want to prove that $|\sum_n t_n f_n(x)| \leq 1$ for all $x \in [0,1]$. Since $f_n \geq 0$ for all $n = 1, 2, \ldots$, we may assume that every t_n is positive. Now, if $x \notin \cup_n(a_n, b_n)$, we have

$$\sum_n t_n f_n(x) \leq \sum_n f_n(x) \leq \sum_n \frac{1}{2^{2n}} \leq \frac{1}{3}.$$

If, on the other hand $x \in (a_{n'}, b_{n'})$ for some $n' \in \mathbb{N}$, then

$$\begin{aligned} \sum_n t_n f_n(x) &\leq f_{n'}(x) + \sum_{n<n'} f_n(x) + \sum_{n>n'} f_n(x) \\ &\leq 1 - \frac{1}{2^{n'}} + \frac{n'-1}{2^{2n'}} + \frac{1}{2^{2n'}} \\ &\leq 1 + \frac{n' - 2^{n'}}{2^{2n'}} \leq 1 - \frac{1}{2^{2n'}} < 1. \end{aligned}$$

These combined yields the right hand side of the inequality (2.1), so the proof is complete. □

A Banach space X contains an *asymptotically isometric copy of* ℓ_1 if it contains a sequence $(x_n)_{n=1}^\infty$ for which there exists a sequence $(\delta_n)_{n=1}^\infty$ in $(0,1)$, decreasing to 0, and such that

$$\sum_{n=1}^m (1-\delta_n)|a_n| \leq \|\sum_{n=1}^m a_n x_n\| \leq \sum_{n=1}^m |a_n|$$

for each finite sequence $(a_n)_{n=1}^m$ in $\mathbb{R}$.

Merging ([DJLT97, Theorem 2]) and [ALNT16, Lemma 2.3] gives us that if either the Banach space X contains an asymptotically isometric copy of c_0 or if X^* is octahedral, then X^* contains an asymptotically isometric copy of ℓ_1. So, we have two ways of proving

Corollary 2.2.7. *$M(\Lambda)^*$ contains an asymptotically isometric copy of ℓ_1.*

Moreover, we have

Corollary 2.2.8. *$M(\Lambda)^{**}$ contains an isometrically isomorphic copy of $L_1[0,1]$.*

Proof. This follows from Corollary 2.2.7 and [DGH00, Theorem 2]. □

Remark 2.2.9 (Added in proof). (a) One of the anonymous referees invited the authors to consider the question whether Müntz spaces also could be octahedral (as $C[0,1]$ is). Here is a preliminary answer: Combine the so called

Clarkson-Erdös-Schwartz theorem (see [GL05, Theorem 6.2.3]) in tandem with a result of P. Wojtaszczyk (see [Wer, Theorem 1]). Then we see that when Λ consists of natural numbers, $M(\Lambda)$ is isomorphic to a subspace of c_0. Since an octahedral space contains a copy of ℓ_1, we have a negative answer for a big class of Müntz spaces.

(b) We have mentioned [ALNT16, Lemma 2.3] as reference for the fact that an octahedral space contains an asymptotically isometric copy of ℓ_1. It has come to our knowledge that this result, even with the same proof, was published earlier by Yamina Yagoub-Zidi, see [YZ13, Proposition 3.3].

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3 On Octahedrality and Müntz spaces

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Published in *Mathematica Scandinavica*, September 2020, volume 126, issue 3, pp. 513–518.

Abstract

We show that every Müntz space can be written as a direct sum of Banach spaces X and Y, where Y is almost isometric to a subspace of c and X is finite dimensional. We apply this to show that no Müntz space is locally octahedral or almost square.

3.1 Introduction

Denote the closed unit ball, the unit sphere, and the dual space of a Banach space X by B_X, S_X, and X^* respectively. Let $\Lambda = (\lambda_i)_{i=0}^{\infty}$, with $\lambda_0 = 0$, be a strictly increasing sequence of non-negative real numbers and let $\Pi(\Lambda) := \text{span}(t^{\lambda_i})_{i=0}^{\infty} \subseteq C[0,1]$, where $C[0,1]$ is the space of real valued continuous functions on $[0,1]$ endowed with the canonical sup-norm $\|\cdot\|_\infty$. We will call $\Lambda = (\lambda_i)_{i=0}^{\infty}$ a *Müntz sequence* and $M(\Lambda) := \overline{\Pi(\Lambda)}$ a *Müntz space* if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. This terminology is justified by Müntz famous theorem from 1914, which says that $\Pi(\Lambda)$ is dense in $C[0,1]$ *if and only if* $\lambda_0 = 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

It is known that a Müntz space $M(\Lambda)$ is isomorphic to a subspace of c_0, provided that the Müntz sequence satisfies the gap condition, i.e. $\inf_{k\in\mathbb{N}}(\lambda_{k+1} - \lambda_k) > 0$ ([GL05, Theorem 9.1.6(c)]). In Section 3.2 we show that all Müntz spaces embed isomorphically into c_0. This is done by showing that $M(\Lambda)$ can be written as a direct sum $X \oplus Y$ where Y is almost isometric to a subspace of c and X is finite dimensional.

Definition 3.1.1. Let X be a Banach space. Then X is

(i) *locally octahedral* (LOH) if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.

(ii) *octahedral* (OH) if for every $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| > 2 - \varepsilon$ for all $i \in \{1, \dots, n\}$.

In Section 3.3 we will show that no Müntz space is OH, answering the question posed in [ALMN17] whether Müntz spaces can be OH. A partial negative answer was given in [ALMN17, Remark 2.9] for Müntz spaces with Müntz sequences consisting only of integers, by combining the Clarkson-Erdös-Schwartz Theorem (see [GL05, Theorem 6.2.3]) with a result of Wojtaszczyk (see [Wern00, Theorem 1]).

Definition 3.1.2. Let X be a Banach space. Then X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $(y_n)_{n=1}^{\infty}$ in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.

(ii) *almost square* (ASQ) if for every $x_1, \dots, x_k \in S_X$ there exists a sequence $(y_n)_{n=1}^{\infty}$ in B_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \dots, k\}$.

Both ASQ and OH are closely related to the area of diameter two properties, which has received intensive attention in the recent years (see for example [BGLPRZ16] and [HLN18] and the references therein). Trivially ASQ implies LASQ and OH implies LOH.

The area of diameter two properties concerns slices of the unit ball, i.e. subsets of the unit ball of the form

$$S(x^*, \varepsilon) := \{x \in B_X : \ x^*(x) > 1 - \varepsilon\},$$

where $x^* \in S_{X^*}$ and $\varepsilon > 0$. Müntz spaces and their diameter two properties were studied in [ALMN17]. Haller, Langemets, Lima and Nadel [HLLN18] pointed out that the proof of [ALMN17, Theorem 2.5] actually shows that, in any $M(\Lambda)$ we have that for every finite family $(S_i)_{i=1}^n$ of slices of $B_{M(\Lambda)}$ and $\varepsilon > 0$, there exist $x_i \in S_i$ and $y \in B_{M(\Lambda)}$, independent of i, such that $x_i \pm y \in S_i$ for every $i \in \{1, \dots, n\}$ and $\|y\| > 1 - \varepsilon$. This property is formally known as *the symmetric strong diameter two property* (SSD2P).

It is known that if a Banach space is ASQ, then it also has the SSD2P. In fact, ASQ is strictly stronger than SSD2P (see [HLLN18, Theorem 2.1d and Example 2.2]). A natural question is therefore if a Müntz space can be ASQ. The results developed in this article will be used to show that this is never the case.

Note that we can exclude the constants and consider the subspace $M_0(\Lambda) := \overline{\text{span}}(t^{\lambda_n})_{n=1}^{\infty}$ of $M(\Lambda)$ and the results of the article still hold true, unless explicitly stated.

We use standard Banach space terminology and notation (e.g. [AK06]), in addition the notation $\|f\|_{[0,a]} := \sup_{x\in[0,a]} |f(x)|$ will be used throughout the paper.

3.2 On embeddings of Müntz spaces

The main results of this article relies on the following results.

Theorem 3.2.1 (Bounded Bernstein's inequality [BE97, Theorem 3.2])**.** *Assume that $1 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then for every $\varepsilon > 0$ there is a constant c_ε such that*

$$\|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon \|p\|_{[0,1]},$$

for all $p \in \Pi(\Lambda)$.

Lemma 3.2.2. *Let V be a subspace of $C[0,1]$ such that each $f \in V$ is differentiable. If for every $\varepsilon > 0$ there exists a $K_\varepsilon \in \mathbb{N}$ such that*

$$\|f'\|_{[0,1-\varepsilon]} \leq K_\varepsilon \|f\|_\infty \tag{3.1}$$

for all $f \in V$, then the Banach space $\overline{V}$ embeds almost isometrically into c.

The proof of Lemma 3.2.2 is almost identical to the proof of [Wern00, Theorem 2], however, we do not require V to be closed, but instead require the inequality (3.1).

Proof. Let $\varepsilon > 0$ and choose a sequence $0 = a_0 < a_1 < \cdots < a_i < \cdots < 1$ converging to 1. For each $a_i \in (0,1)$ there exists $K_i > 0$, depending on a_i such that

$$\|f'\|_{[0,a_i]} \leq K_i \|f\|_\infty \text{ for all } f \in V$$

Pick points $0 = s_0 < s_1 < \cdots < s_{n_1} = a_1 < s_{n_1+1} < \cdots < s_{n_2} = a_2 < \cdots$, in such a way that

$$s_{j+1} - s_j \leq \frac{\varepsilon}{K_{i+1}} \text{ for } n_i \leq j < n_{i+1}.$$

Define the operator $J_\varepsilon : \overline{V} \to c$ by $J_\varepsilon(f) = (f(s_n))_n$, thus J_ε is well-defined by continuity of $f \in \overline{V}$. As $||J_\varepsilon f|| = \sup_{n\in\mathbb{N}} |f(s_n)| \leq ||f||_\infty$, for all $f \in \overline{V}$, we have that $\|J_\varepsilon\| \leq 1$. For any $f \in \overline{V}$ let (f_k) be a sequence in V converging uniformly to

f. Let $\delta > 0$ and find $N \in \mathbb{N}$ such that $\|f - f_N\|_\infty < \delta$. Then, for any $s \in [0,1)$, we have $a_i \leq s < a_{i+1}$ for some $i \in \mathbb{N}$. Let $s_m \in [a_i, a_{i+1}]$ be such that $|s - s_m| \leq \frac{\varepsilon}{K_{i+1}}$. Then

$$\begin{aligned} |f(s)| \leq |f_N(s)| + \delta &\leq |f_N(s) - f_N(s_m)| + |f_N(s_m)| + \delta \\ &\leq \sup_{a_i \leq t \leq a_{i+1}} |f'_N(t)||s - s_m| + \|J_\varepsilon f_N\| + \delta \\ &\leq \|f_N\|_\infty K_{i+1} \frac{\varepsilon}{K_{i+1}} + \|J_\varepsilon f_N\| + \delta \\ &\leq \|f_N\|_\infty \varepsilon + \|J_\varepsilon f_N\| + \delta \\ &\leq (\|f\|_\infty + \delta)\varepsilon + (\|J_\varepsilon f\| + \delta) + \delta \end{aligned}$$

and therefore

$$(1 - \varepsilon)\|f\|_\infty - \delta(\varepsilon + 2) \leq \|J_\varepsilon f\|.$$

Since δ was arbitrary we conclude that

$$(1 - \varepsilon)\|f\|_\infty \leq \|J_\varepsilon f\| \leq \|f\|_\infty,$$

completing the proof. □

Combining Theorem 3.2.1 and Lemma 3.2.2, we arrive at the following proposition.

Proposition 3.2.3. *Let Λ be a Müntz sequence with $\lambda_1 \geq 1$. Then the associated Müntz space $M(\Lambda)$ is almost isometric to a subset of c. That is, for every $\varepsilon > 0$ there is an operator $J_\varepsilon : M(\Lambda) \to c$ such that*

$$(1 - \varepsilon)\|f\|_{[0,1]} \leq \|J_\varepsilon f\| \leq \|f\|_{[0,1]}.$$

We will need the following lemma for the coming theorem.

Lemma 3.2.4. *Let $Z = \overline{span}(z_i)_{i \in \mathbb{N}}$ and let $N \in \mathbb{N}$. If $Y = \overline{span}(z_i)_{i > N}$ then $Z/Y = \operatorname{span}(\pi(z_i))_{i \leq N}$, where $\pi : Z \to Z/Y$ is the quotient map. Consequently Z/Y has finite dimension and $Z = X \oplus Y$ where $X = \operatorname{span}(x_i)_{i \leq N}$.*

Remark 3.2.5. For every $N \in \mathbb{N}$ we have that $\overline{\operatorname{span}}(t^{\lambda_i})_{i \geq N}$ is a finite codimensional subspace of $M(\Lambda)$.

By combining Proposition 3.2.3 and Lemma 3.2.4 we obtain the following result.

Theorem 3.2.6. *Every Müntz space $M(\Lambda)$ can be written as $X \oplus Y$ where X is finite dimensional and Y is almost isometric to a subspace of c.*

Corollary 3.2.7. *Every Müntz space $M(\Lambda)$ embeds isomorphically into c_0.*

Remark 3.2.8. From [GL05, Theorem 10.4.4] it is known that no Müntz space of dimension greater than 2 is polyhedral. However, since c_0 is polyhedral ([Klee60, Theorem 4.7]), it follows that any Müntz space can be renormed to be polyhedral.

3.3 On octahedrality and almost squareness of Müntz spaces

The results from Section 3.2 will now be used to derive some results concerning Müntz spaces. $M(\Lambda)^*$ is separable by Corollary 3.2.7, we therefore easily answer the question posed in [ALMN17]. In fact we show more.

Theorem 3.3.1. *No Müntz space $M(\Lambda)$ is LOH.*

Proof. Since $M(\Lambda)^*$ is separable, we can combine [Bour83, Theorem 4.1.3] with [Bour83, Theorem 4.2.13] to see that there exist slices $S(x, \varepsilon)$ of the unit ball of $M(\Lambda)^*$ of arbitrarily small diameter, where x can be taken from $M(\Lambda)$. By [HLP15, Theorem 3.1] this is equivalent to $M(\Lambda)$ failing to be LOH, as claimed. □

We finish this article by showing that $M_0(\Lambda)$ fails to be ASQ for any Müntz sequence Λ. Note that $M(\Lambda)$ is trivially not LASQ, just consider the constant function 1. First we show that even more is true for some spaces $M_0(\Lambda)$.

Proposition 3.3.2. *No Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$ is LASQ.*

Proof. Let Λ be a Müntz sequence with $\lambda_1 \geq 1$ and $M_0(\Lambda)$ be the associated Müntz space. Choose some $x \in (0, 1)$. By Theorem 3.2.1 there is a $c \in \mathbb{N}$ such that $\|f'\|_{[0,x]} \leq c$ for all $f \in B_{\Pi(\Lambda)}$. Let $a = \min(\frac{1}{2c}, x)$ and observe that

$$\sup_{f \in B_{\Pi(\Lambda)}} \|f\|_{[0,a]} \leq \frac{1}{2}$$

since

$$|f(t)| = |f(t) - f(0)| \leq \|f'\|_{[0,a]} \cdot |t - 0| \leq c \cdot \frac{1}{2c} = \frac{1}{2}.$$

Recall from [ALV16, Theorem 2.1] that $M_0(\Lambda)$ is LASQ if and only if for every $g \in S_{M(\Lambda)}$ and $\varepsilon > 0$ there exists $h \in S_{M(\Lambda)}$ such that $\|g \pm h\| \leq 1 + \varepsilon$. We claim that no such h exists for $g = t^{\lambda_1}$. Indeed, if $0 < \varepsilon < a^{\lambda_1}/2$ and $h \in S_{\Pi(\Lambda)}$ is such that

$\|t^{\lambda_1} \pm h\| \leq 1 + \varepsilon$, then $|h(t)| < 1 - \varepsilon$ for $t \geq a$ as $t^{\lambda_1} > 2\varepsilon$ for $t \geq a$. Thus, h must attain its norm on the interval $[0, a]$, contradicting our observation. As $\Pi((\lambda_n)_{n=1}^{\infty})$ is dense in $M_0(\Lambda)$, we conclude that $M_0(\Lambda)$ is not LASQ. □

Proposition 3.3.3. *No Müntz space $M_0(\Lambda)$ is ASQ.*

Proof. Combining Lemma 3.2.4 with Proposition 3.3.2 shows that every Müntz space $M_0(\Lambda)$ has a subspace of finite codimension which is not ASQ. By [Abra15, Theorem 3.6] no Müntz space can be ASQ. □

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4 Daugavet- and delta-points in Banach spaces with unconditional bases

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To appear in *Transactions of the American Mathematical Society* (2021)

Abstract

We study the existence of Daugavet- and delta-points in the unit sphere of Banach spaces with a 1-unconditional basis. A norm one element x in a Banach space is a Daugavet-point (resp. delta-point) if every element in the unit ball (resp. x itself) is in the closed convex hull of unit ball elements that are almost at distance 2 from x. A Banach space has the Daugavet property (resp. diametral local diameter two property) if and only if every norm one element is a Daugavet-point (resp. delta-point). It is well-known that a Banach space with the Daugavet property does not have an unconditional basis. Similarly spaces with the diametral local diameter two property do not have an unconditional basis with suppression unconditional constant strictly less than 2.

We show that no Banach space with a subsymmetric basis can have delta-points. In contrast we construct a Banach space with a 1-unconditional basis with delta-points, but with no Daugavet-points, and a Banach space with a 1-unconditional basis with a unit ball in which the Daugavet-points are weakly dense.

4.1 Introduction

Let X be a Banach space with unit ball B_X, unit sphere S_X, and topological dual X^*. For $x \in S_X$ and $\varepsilon > 0$ let $\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}$. We say that X has the

(i) *Daugavet property* if for every $x \in S_X$ and every $\varepsilon > 0$ we have $B_X = \overline{\text{conv}}\Delta_\varepsilon(x)$;

(ii) *diametral local diameter two property* if for every $x \in S_X$ and every $\varepsilon > 0$ we have $x \in \overline{\text{conv}}\Delta_\varepsilon(x)$.

In [Kad96, Corollary 2.3] Kadets proved that any Banach space with the Daugavet property fails to have an unconditional basis (see also [Wer01, Proposition 3.1]). These arguments are probably the easiest known proofs of the absence of unconditional bases in the classical Banach spaces $C[0,1]$ and $L_1[0,1]$. The diametral local diameter two property was named and studied in [BGLPRZ18], but it was first introduced in [IK04] under the name *space with bad projections*. (See the references in [IK04] for previous unnamed appearances of this property.) Using the characterizations in [IK04] we see that if a Banach space with the diametral local diameter two property has an unconditional basis, then the unconditional suppression basis constant is at least 2. But note that we do not know of any Banach space with an unconditional basis and the diametral local diameter two property.

In the present paper we study pointwise versions of the Daugavet property and the diametral local diameter two property in spaces with 1-unconditional bases.

Definition 4.1.1. Let X be a Banach space and let $x \in S_X$. We say that x is

(i) a *Daugavet-point* if for every $\varepsilon > 0$ we have $B_X = \overline{\operatorname{conv}}\Delta_\varepsilon(x)$;

(ii) a *delta-point* if for every $\varepsilon > 0$ we have $x \in \overline{\operatorname{conv}}\Delta_\varepsilon(x)$.

Daugavet-points and delta-points were introduced in [AHLP20]. For the spaces $L_1(\mu)$, for preduals of such spaces, and for Müntz spaces these notions are the same [AHLP20, Theorems 3.1, 3.7, and 3.13]. However, $C[0,1] \oplus_2 C[0,1]$ is an example of a space with the diametral local diameter two property, but with no Daugavet-points [AHLP20, Example 4.7]. Stability results for Daugavet- and delta-points in absolute sums of Banach spaces was further studied in [HPV21].

In Section 4.2 we consider Banach spaces with 1-unconditional bases and study a family of subsets of the support of a vector x. We find properties of these subsets that are intimately linked to x not being a delta-point. Quite general results are obtained in this direction. We apply these results to show that Banach spaces with subsymmetric bases (these include separable Lorentz and Orlicz sequence spaces) always fail to contain delta-points.

In Section 4.3 we construct a Banach space with a 1-unconditional basis which contains a delta-point, but contain no Daugavet-points. The example is a Banach space of the type $h_{\mathcal{A},1}$ generated by an adequate family of subsets of a binary tree. The norm of the space is the supremum of the ℓ_1-sum of branches in the binary tree.

In Section 4.4 we modify slightly the binary tree from Section 4.3 and the associated adequate family, to obtain an $h_{\mathcal{A},1}$ space with some remarkable properties: It has Daugavet-points; the Daugavet-points are even weakly dense in the unit ball; the diameter of every slice of the unit ball is two, but is has relatively weakly open subsets of the unit ball of arbitrary small diameter.

Finally, let us also remark that the examples in both Section 4.3 and Section 4.4 contain isometric copies of c_0 and ℓ_1. Both the ℓ_1-ness of the branches and c_0-ness of antichains in the binary tree play an important role in our construction of Daugavet- and delta-points in these spaces (see e.g. Theorems 4.3.1 and 4.4.2, and Corollary 4.4.3).

4.2 1-unconditional bases and the sets $M(x)$

The main goal of this section is to prove that Banach spaces with a subsymmetric basis fail to have delta-points. Before we start this mission, let us point out some results and concepts that we will need. First some characterizations of Daugavet- and delta-points that we will frequently use throughout the paper.

Recall that a *slice* of the unit ball B_X of a Banach space X is a subset of the form

$$S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > \|x^*\| - \varepsilon\},$$

where $x^* \in X^*$ and $\varepsilon > 0$.

Proposition 4.2.1. *[AHLP20, Lemma 2.3] Let X be a Banach space and $x \in S_X$. The following assertions are equivalent:*

(i) x is a Daugavet-point;

(ii) for every slice S of B_X and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

Proposition 4.2.2. *[AHLP20, Lemma 2.2] Let X be a Banach space and $x \in S_X$. The following assertions are equivalent:*

(i) x is a delta-point;

(ii) for every slice S of B_X with $x \in S$ and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

Let X be a Banach space. Recall that a Schauder basis $(e_i)_{i\in\mathbb{N}}$ of X is called *unconditional* if for every $x \in X$ its expansion $x = \sum_{i\in\mathbb{N}} x_i e_i$ converges unconditionally. If, moreover, $\|\sum_{i\in\mathbb{N}} \theta_i x_i e_i\| = \|\sum_{i\in\mathbb{N}} x_i e_i\|$ for any $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$ and any sequence of signs $(\theta_i)_{i\in\mathbb{N}}$, then $(e_i)_{i\in\mathbb{N}}$ is called 1-unconditional. A Schauder basis is called *subsymmetric*, or 1-*subsymmetric*, if it is unconditional and $\|\sum_{i\in\mathbb{N}} \theta_i x_i e_{k_i}\| = \|\sum_{i\in\mathbb{N}} x_i e_i\|$ for any $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$, any sequence of signs $(\theta_i)_{i\in\mathbb{N}}$, and any infinite increasing sequence of naturals $(k_i)_{i\in\mathbb{N}}$. Trivially a subsymmetric basis is 1-unconditional. In the following we will assume that the basis $(e_i)_{i\in\mathbb{N}}$ is normalized, i.e. $\|e_i\| = 1$ for all $i \in \mathbb{N}$. With $(e_i^*)_{i\in\mathbb{N}}$ we denote the conjugate in X^* to the basis $(e_i)_{i\in\mathbb{N}}$. Clearly $(e_i^*)_{i\in\mathbb{N}}$ is a 1-unconditional basic sequence whenever $(e_i)_{i\in\mathbb{N}}$ is. When studying Daugavet-points or delta-points in a Banach space X with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ we can restrict our investigation to the positive cone K_X generated by the basis, where

$$K_X = \left\{ x = \sum_{i\in\mathbb{N}} x_i e_i : x_i \geq 0 \right\} = \{x \in X : e_i^*(x) \geq 0\}.$$

The reason for this is that for every sequence of signs $\theta = (\theta_i)_{i\in\mathbb{N}}$ the operator $T_\theta : X \to X$ defined by $T_\theta(\sum_{i\in\mathbb{N}} x_i e_i) = \sum_{i\in\mathbb{N}} \theta_i x_i e_i$ is a linear isometry. Hence $x = \sum_{i\in\mathbb{N}} x_i e_i$ is a Daugavet-point (resp. delta-point) if and only if $|x| = \sum_{i\in\mathbb{N}} |x_i| e_i$ is.

The following result is well-known.

Proposition 4.2.3. *Let X be a Banach space with a 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $\sum_{i\in\mathbb{N}} b_i e_i$ is convergent and $|a_i| \leq |b_i|$ for all i, then $\sum_{i\in\mathbb{N}} a_i e_i$ is convergent and*

$$\Big\|\sum_{i\in\mathbb{N}} a_i e_i\Big\| \leq \Big\|\sum_{i\in\mathbb{N}} b_i e_i\Big\|.$$

Moreover $\|P_A\| = 1$ where, for $A \subset \mathbb{N}$, P_A is the projection defined by

$$P_A(\sum_{i\in\mathbb{N}} x_i e_i) = \sum_{i\in A} x_i e_i.$$

From this we immediately get a fact that will be applied several times throughout the paper.

Fact 4.2.4. Let X be a Banach space with a 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ and let $x, y \in X$ and $E \subset \mathbb{N}$. Then the following holds.

- If $|x_i| \leq |y_i|$ and $\operatorname{sgn} x_i = \operatorname{sgn} y_i$ for all $i \in E$, then $\|y - P_E x\| \leq \|y\|$.

The upshot of Fact 4.2.4 is that it can be used to find an upper bound for the distance between $x \in S_X$ and elements in a given subset of the unit ball. Indeed, suppose we can find $E \subseteq \mathbb{N}$, $\eta > 0$ and a subset S of the unit ball such that $\|x - P_E x\| < 1 - \eta$ and the assumption in Fact 4.2.4 holds for any $y \in S$. Then

$$\|x - y\| \leq \|x - P_E x\| + \|y - P_E x\| < 2 - \eta.$$

If such a set S is a slice (resp. a slice containing x), then x cannot be a Daugavet-point (resp. delta-point). We will see in Theorem 4.2.17 that any unit sphere element in a space with a subsymmetric basis, is contained in a slice of the above type. Our tool to investigate the existence of slices of this type in a Banach space with a 1-unconditional basis, are certain families of subsets of the support of the elements in the space.

Remark 4.2.5. If only the moreover part of Proposition 4.2.3 holds, then the basis is called 1-suppression unconditional. In this case the conclusion of Proposition 4.2.3 still holds if $\operatorname{sgn} a_i = \operatorname{sgn} b_i$, for all i. This is all that is needed in Fact 4.2.4. Similarly, one can check that all the results about 1-unconditional bases in the rest of this section also holds for a Banach space X with a 1-suppression unconditional basis.

Definition 4.2.6. For any Banach space X with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ and for $x \in X$, define

$$M(x) := \{A \subseteq \mathbb{N} : \|P_A x\| = \|x\|, \|P_A x - x_i e_i\| < \|x\|, \text{ for all } i \in A\},$$

$$M^{\mathcal{F}}(x) := \{A \in M(x) : |A| < \infty\},$$

and

$$M^{\infty}(x) := \{A \in M(x) : |A| = \infty\}.$$

We can think of $M(x)$ as a collection of minimal "norm-giving" subsets of the support of x. If for example $X = c_0$ and $x \in c_0$, then $M(x) = \{\{i\} : |x_i| = \|x\|\}$ while if $X = \ell_p$, $1 \leq p < \infty$ and $x \in X$, then $M(x) = \{\operatorname{supp}(x)\}$.

Our first observation about the families $M(x)$ is that they are always non-empty.

Lemma 4.2.7. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. Then $M(x) \neq \emptyset$ for all $x \in X$.*

Proof. Let $x \in X$. Either $A_0 := \text{supp}(x) \in M(x)$ or there exists a smallest $n_1 \in A_0$ such that if we define $A_1 = A_0 \setminus \{n_1\}$, then $\|P_{A_1}x\| = \|x\|$ and

$$\|P_{A_1}x - x_j e_j\| < \|x\| \text{ for all } j \in A_0 \cap \{1, \dots, n_1 - 1\}.$$

Suppose we have found $n_1 < \cdots < n_{k-1}$ such that $A_{k-1} = A_{k-2} \setminus \{n_{k-1}\}$ satisfies $\|P_{A_{k-1}}x\| = \|x\|$ and $\|P_{A_{k-1}}x - x_j e_j\| < \|x\|$ for all $j \in A_{k-1} \cap \{1, \dots, n_{k-1} - 1\}$. Then either $A_{k-1} \in M(x)$ or there exists a smallest integer n_k greater than n_{k-1} such that $A_k = A_{k-1}(x) \setminus \{n_k\}$ satisfies $\|P_{A_k}x\| = \|x\|$ and

$$\|P_{A_k}x - x_j e_j\| < \|x\| \text{ for all } j \in A_k \cap \{1, \dots, n_k - 1\}.$$

Either this process terminates and $A_k \in M(x)$, or we get a set $N = \{n_i\}_{i=1}^{\infty}$. Let $A = \bigcap_k A_k = \text{supp}(x) \setminus N$ and note that $\|P_A x\| = \|x\|$. If $j \in A$, find k such that $j < n_k$, then by 1-unconditionality

$$\|P_A x - x_j e_j\| \leq \|P_{A_k}x - x_j e_j\| < \|x\|$$

and $A \in M(x)$. □

Our next goal is to prove that certain classes of subsets of $M^{\mathcal{F}}(x)$ and $M^{\infty}(x)$ are finite (see Lemma 4.2.10 below). We will use the next result as a stepping stone. In the proof, and throughout the paper, we will assume that the sets $A = \{a_1, a_2, \dots\} \in M(x)$ are ordered so that $a_1 < a_2 < \cdots < a_n < \cdots$, and we will use $A(n)$ to denote the set $\{a_1, \dots, a_n\}$.

Lemma 4.2.8. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i \in \mathbb{N}}$. If $x \in X$, then for every $n \in \mathbb{N}$,*

(i) $|\{A(n) : A \in M(x),\ |A| > n\}| < \infty$*;*

(ii) $|\{A \in M(x) : |A| \leq n\}| < \infty$.

In particular, $\left|\bigcup_{D \in M^{\infty}(x)} \{D(n)\}\right| < \infty$.

Proof. Let us prove (i) inductively. For $k \in \mathbb{N}$, let $R_k = I - P_{N_k}$, where $N_k = \{1, \dots, k\}$. For $n = 1$ the result follows from $\|R_k x\| \to 0$.

Now assume that $|\{A(n-1) : A \in M(x),\ |A| > n-1\}| < \infty$, and let $s_{n-1} := \max\left\{\left\|P_{A(n-1)}x\right\| : A \in M(x), |A| > n-1\right\} < \|x\|$. Find $k \in \mathbb{N}$ such that $\|R_k x\| < \|x\| - s_{n-1}$. Then by the triangle inequality, it follows that $\max A(n) \leq k$ for all $A \in M(x)$ with $|A| > n$.

For (ii), let $A \in M(x)$ with $|A| = n$. Then $\left\|P_{A(n-1)}x\right\| \leq s_{n-1}$, and thus $\max A \leq k$, where as above $k \in \mathbb{N}$ is such that $\|R_k x\| < \|x\| - s_{n-1}$. □

In order to find the sets $E \subseteq \mathbb{N}$ mentioned in the remarks following Fact 4.2.4 we need the following families of subsets of $M(x)$.

Definition 4.2.9. Let X have 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. Let $x \in S_X$ and define

$$\mathcal{F}_n(x) := \left\{A \in M^{\mathcal{F}}(x) : A \cap D(n) \neq D(n), \text{ for all } D \in M^\infty(x)\right\},$$
$$\mathcal{G}_n(x) := \mathcal{F}_n(x) \cup \bigcup_{D\in M^\infty(x)} \{D(n)\},$$
$$\mathcal{E}_n(x) := \left\{E \subset \bigcup_{A\in\mathcal{G}_n} A : E \cap A \neq \emptyset, \text{ for all } A \in \mathcal{G}_n\right\}.$$

If it is clear from the context what element x we are considering, we will simply denote these sets by $\mathcal{F}_n, \mathcal{G}_n$, and $\mathcal{E}_n$.

It is pertinent with a couple of comments about these families of sets. Trivially, if $M^\infty(x) = \emptyset$, then $\mathcal{G}_n = \mathcal{F}_n = M(x)$ for all $n \in \mathbb{N}$. We can think of the elements of $\mathcal{E}_n$ as essential for the norm of x, i.e. $\|x - P_E x\| < \|x\|$ for all $E \in \mathcal{E}_n$. According to Lemma 4.2.11 below the drop in norm is also uniformly bounded away from 0. The main reason for this is that $\mathcal{F}_n$ and $\mathcal{E}_n$ are finite for all $n \in \mathbb{N}$. We will prove this now.

Lemma 4.2.10. *Let X have 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $x \in S_X$, then for all $n \in \mathbb{N}$,*

(i) $|\mathcal{F}_n| < \infty$*;*

(ii) $|\mathcal{E}_n| < \infty$.

In particular, if $M^\infty(x) = \emptyset$, then $|M(x)| < \infty$.

Proof. (i). There exists $N \in \mathbb{N}$ such that $\max_{D\in M^\infty(x)} D(n) \leq N$ by Lemma 4.2.8.

Assume for contradiction that $|\mathcal{F}_n| = \infty$. Then there exists a sequence $(A_k) \subset \mathcal{F}_n$ such that $|A_k| \geq k$. By compactness of $\{0,1\}^{\mathbb{N}}$ and passing to a subsequence if necessary, we may assume that $A_k \to A \in \mathbb{N}$ pointwise and $A \cap \{1,\ldots,N\} = A_k \cap \{1,\ldots,N\}$ for all k. In particular $\|P_A x\| = 1$. By Lemma 4.2.7, there exists $B \subseteq A$, such that $B \in M(P_A x) \subseteq M(x)$. Since $A \cap \{1,\ldots,N\} = A_k \cap \{1,\ldots,N\}$, we have $|B| < \infty$ by definition of $\mathcal{F}_n$. Since B is finite $A_k \cap B$ is eventually constant. Thus for some $k \in \mathbb{N}$ we have $B \subsetneq A_k \in M(x)$, a contradiction.

Finally, (ii) follows from (i) and Lemma 4.2.8. □

With the knowledge that the cardinality of $\mathcal{E}_n$ is finite for every $n \in \mathbb{N}$, we now obtain the following result.

Lemma 4.2.11. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $x \in S_X$, then*

(i) $\|x - P_E x\| < 1$ if $E \cap A \neq \emptyset$ for all $A \in M(x)$;

(ii) for any $n \in \mathbb{N}$ there exists $\gamma_n > 0$ such that

$$\max_{E\in\mathcal{E}_n} \|x - P_E x\| = 1 - \gamma_n.$$

Proof. (i). Assume that $E \subseteq \mathbb{N}$ with $E \cap A \neq \emptyset$ for all $A \in M(x)$ such that $\|x - P_E x\| = 1$. By Lemma 4.2.7 there exists $B \in M(x-P_E x)$. But $M(x-P_E x) \subseteq M(x)$ since $\|x - P_E x\| = 1$ and this gives us the contradiction $B \cap E = \emptyset$.

Any $E \in \mathcal{E}_n$ satisfies $E \cap A \neq \emptyset$ for all $A \in M(x)$ and $\mathcal{E}_n$ is finite, so (ii) follows from (i). □

Let X be a Banach space and $x \in S_X$. If x is a delta-point, then for every slice S with $x \in S$, we have that x is at one end of a line segment in S with length as close to 2 as we want. Suppose we replace the slice S with a non-empty relatively weakly open subset W of B_X with $x \in W$. If X has the Daugavet property, then x is at one end of a line segment in W with length as close to 2 as we want ([Shv00, Lemma 3]). Next we show that this is never the case if X has a 1-unconditional basis.

Proposition 4.2.12. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $x \in S_X$, then there exist $\delta > 0$ and a relatively weakly open subset W, with $x \in W$, such that $\sup_{y\in W} \|x - y\| < 2 - \delta$.*

Proof. Assume that $x \in S_X \cap K_X$. Let $E = \bigcup_{A\in M(x)} A(1)$. By Lemma 4.2.11 there exists $\gamma_1 > 0$ such that $\max_{F\in\mathcal{E}_1} \|x - P_F x\| = 1 - \gamma_1$. Let $\delta = \gamma_1/2$.

Let $W = \left\{y \in B_X : \left|e_i^*(x - y)\right| < \min_{k\in E} \frac{x_k}{2}, i \in E\right\}$. Then $x \in W$, and if $y \in W$, then $y_i \geq \frac{x_i}{2} > 0$ for all $i \in E$. Thus if $y \in W$ we have

$$\{i \in \mathbb{N} : y_i \geq \frac{x_i}{2}\} \cap E = E \in \mathcal{E}_1.$$

For any $y \in W$, we get that

$$\|x - y\| \leq \left\|\frac{x}{2}\right\| + \left\|\frac{x}{2} - P_E\frac{x}{2}\right\| + \left\|P_E\frac{x}{2} - y\right\| < 2 - \delta,$$

and we are done. □

Let us remark a fun application of the above proposition.

Remark 4.2.13. Let K be an infinite compact Hausdorff space. Then $C(K)$ does not have a 1-unconditional (or a 1-suppression unconditional) basis.

Let f be a function which attains its norm on a limit point of K. Arguing similarly as in [AHLP20, Theorem 3.4] we may find a sequence of norm one functions g_k with distance as close to 2 as we want from f that converge pointwise, and thus weakly, to f. The conclusion follows from Proposition 4.2.12.

The next result is the key ingredient in our proof that there are no delta-points in Banach spaces with subsymmetric bases. Its proof draws heavily upon Lemma 4.2.11.

Lemma 4.2.14. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ and let $x \in S_X$. Assume that there exists a slice $S(x^*,\delta)$, an $n \in \mathbb{N}$ and some $\eta > 0$ such that*

(i) $x \in S(x^*,\delta)$,

(ii) $y \in S(x^*,\delta)$ *implies that*

$$\{i : |y_i| > \eta|x_i|,\ \operatorname{sgn} y_i = \operatorname{sgn} x_i\} \cap D(n) \neq \emptyset$$

for all $D \in M^\infty(x)$.

Then x is not a delta-point.

Proof. Assume that $x \in S_X \cap K_X$. Now for each $A \in \mathcal{F}_n$ find $x_A^* \in S_{X^*}$ such that $x_A^*(P_A x) = 1$ with $x_A^*(e_i) = 0$ for all $i \notin A$, and $x_A^*(e_i) > 0$ for all $i \in A$. Let $z^* = \frac{1}{|\mathcal{F}_n|+1}\left(\sum_{A\in\mathcal{F}_n} x_A^* + x^*\right)$. Then $z^* \in B_{X^*}$ and

$$\|z^*\| \geq z^*(x) > \frac{|\mathcal{F}_n| + 1 - \delta}{|\mathcal{F}_n| + 1} = 1 - \frac{\delta}{|\mathcal{F}_n| + 1}.$$

For any $y \in S(z^*, \|z^*\| - 1 + \frac{\delta}{|\mathcal{F}_n|+1})$, we get that

$$1 - \frac{\delta}{|\mathcal{F}_n| + 1} < \frac{1}{|\mathcal{F}_n| + 1}\left(\sum_{A\in\mathcal{F}_n} x_A^*(y) + x^*(y)\right) \leq \frac{|\mathcal{F}_n| + x^*(y)}{|\mathcal{F}_n| + 1}.$$

Solving for $x^*(y)$ we get that

$$1 - \delta < x^*(y),$$

and similarly $1 - \delta < x_A^*(y)$. Thus, if $0 < \eta < 1 - \delta$,

$$F := \{i : y_i \geq \eta x_i\} \bigcap \left(\bigcup_{E\in\mathcal{G}_n} E\right) \in \mathcal{E}_n.$$

For any $y \in S(z^*, \|z^*\| - 1 + \frac{\delta}{|\mathcal{F}_n|+1})$ we now get from Lemma 4.2.11 that

$$\begin{aligned}\|x - y\| &\leq \|x - \eta P_F x\| + \|\eta P_F x - y\| \\ &\leq \eta \|x - P_F x\| + (1-\eta)\|x\| + 1 \\ &\leq \eta \max_{E \in \mathcal{E}_n} \|x - P_E x\| + 2 - \eta \\ &\leq 2 - \eta\gamma_n < 2.\end{aligned}$$

□

If $x \in S_X$ with $M^\infty(x) = \emptyset$ in the above lemma, then any slice $S(x^*, \delta)$ containing x trivially satisfies Lemma 4.2.14 (ii). We record this in the following proposition.

Proposition 4.2.15. *Let X be a Banach space with 1-unconditional basis and let $x \in S_X$. If $M^\infty(x) = \emptyset$, then x is not a delta-point.*

We will also need the following lemma.

Lemma 4.2.16. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $x \in K_X$, then for every $A \in M(x)$ and every $t > 0$ we have $\|P_A x + te_i\| > \|x\|$ for all $i \in A$.*

Proof. Let $x \in K_X$, $A \in M(x)$ and $t > 0$. Since $i \in A$ and $A \in M(x)$ we have $x_i > 0$ and $\|P_A x - x_i\| < \|x\| = \|P_A x\|$. Put $\lambda = x_i/(t + x_i)$. Then $0 < \lambda < 1$ and $P_A x = \lambda(P_A x + te_i) + (1-\lambda)(P_A x - x_i e_i)$, so

$$\begin{aligned}\|x\| = \|P_A x\| &\leq \lambda\|P_A x + te_i\| + (1-\lambda)\|P_A x - x_i e_i\| \\ &< \lambda\|P_A x + te_i\| + (1-\lambda)\|P_A x\| \\ &= \lambda\|P_A x + te_i\| + (1-\lambda)\|x\|,\end{aligned}$$

and the conclusion follows. □

Finally it is time to cash in some dividends and prove the main result of this section.

Theorem 4.2.17. *If X has subsymmetric basis $(e_i)_{i\in\mathbb{N}}$, then X has no delta-points.*

Proof. Assume $x \in S_X \cap K_X$. By Proposition 4.2.15 we may assume that $M^\infty(x) \neq \emptyset$. Let $s := \max\{n : x_n = \max_i x_i\}$. We first show that $s \in A$ for all $A \in M^\infty(x)$.

For contradiction assume that there exists $A = \{a_1, a_2, \ldots\} \in M^\infty(x)$ with $s \notin A$. Let $a_0 = 0$ and $j \in \mathbb{N}$ be such that $a_{j-1} < s < a_j$. Let $t > 0$ such that $x_s = x_{a_j} + t$

and let A_s be A with a_j replaced by s. Using that $(e_i)_{i\in\mathbb{N}}$ is subsymmetric and Lemma 4.2.16 we get

$$\begin{aligned} 1 \geq \|P_{A_s}x\| &= \left\| \sum_{i\neq j} x_{a_i}e_i + (x_{a_j}+t)e_j \right\| \\ &= \left\| \sum_{i\in\mathbb{N}} x_{a_i}e_{a_i} + te_{a_j} \right\| = \|P_A x + te_{a_j}\| > 1 \end{aligned}$$

a contradiction.

If we let $n = s$, then $s \in D(n)$ for all $D \in M^\infty(x)$, and the slice $S(e_s^*, 1 - \frac{x_s}{2})$ and $\eta = \frac{1}{2}$ satisfies the criteria in Lemma 4.2.14 and we are done. □

In the proof above we saw that if X has a subsymmetric basis, then for any $x \in S_X$ either $M^\infty(x) = \emptyset$ or all $A \in M^\infty(x)$ has a common element. In the case X has a 1-symmetric basis we can say a lot about the sets $M(x)$ for any given $x \in S_X$.

Recall that a Schauder basis $(e_i)_{i\in\mathbb{N}}$ is called 1-*symmetric* if it is unconditional and $\|\sum_{i\in\mathbb{N}} \theta_i x_i e_{\pi(i)}\| = \|\sum_{i\in\mathbb{N}} x_i e_i\|$ for any $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$, any sequence of signs $(\theta_i)_{i\in\mathbb{N}}$, and any permutation π of $\mathbb{N}$. A 1-symmetric basis is subsymmetric [LT77, Proposition 3.a.3].

Proposition 4.2.18. *Let X be a Banach space with 1-symmetric basis $(e_i)_{i\in\mathbb{N}}$ and let $x \in S_X$.*

(i) If $M^\infty(x) \neq \emptyset$, then $M(x) = \{\text{supp}(x)\}$;

(ii) If $M^\infty(x) = \emptyset$ and $A, B \in M(x)$, then $|A| = |B|$ and x is constant on $A \triangle B$.

Proof. Assume that $x \in S_X \cap K_X$.

(i). Let $A \in M^\infty(x)$ and $x_l \in \text{supp}(x) \setminus A$. Since $|A| = \infty$, there exists $k \in A$ and $t > 0$ with $x_k + t = x_l$. Using that $(e_i)_{i\in\mathbb{N}}$ is 1-symmetric and Lemma 4.2.16 we get

$$1 \geq \left\|P_{A\setminus\{k\}}x + x_l e_l\right\| = \left\|P_{A\setminus\{k\}}x + x_l e_k\right\| = \|P_A x + te_k\| > 1,$$

a contradiction.

(ii). Suppose that x is not constant on $A \triangle B$ and let $k, l \in A \triangle B$ with $x_k \neq x_l$, say $k \in A, l \in B$, and $x_k < x_l$. Then argue as in (i) to get a contradiction, so x is constant on $A \triangle B$. As x is constant on $A \triangle B$, we cannot have $|A| < |B|$ since then a subset of B would be in $M(x)$ contradicting the definition of $M(x)$. □

4.3 A space with 1-unconditional basis and delta-points

In this section we will prove the following theorem.

Theorem 4.3.1. *There exists a Banach space $X_{\mathfrak{B}}$ with 1-unconditional basis, such that*

(i) $X_{\mathfrak{B}}$ has a delta-point;

(ii) $X_{\mathfrak{B}}$ does not have Daugavet-points.

Before giving a proof of the theorem we will need some notation. By definition, a *tree* is a partially ordered set $(\mathcal{T}, \preceq)$ with the property that, for every $t \in \mathcal{T}$, the set $\{s \in \mathcal{T} : s \preceq t\}$ is well ordered by $\preceq$. In any tree we use normal interval notation, so that for instance a *segment* is $[s, t] = \{r \in \mathcal{T} : s \preceq r \preceq t\}$. If a tree has only one minimal member, it is said to be *rooted* and the minimal member is called the *root* of the tree and is denoted $\emptyset$. We have $\emptyset \preceq t$ for all $t \in \mathcal{T}$. We say that t is an *immediate successor* of s if $s \prec t$ and the set $\{r \in \mathcal{T} : s \prec r \prec t\}$ is empty. The set of immediate successors of s we denote with s^+. A sequence $B = \{t_n\}_{n=0}^{\infty}$ is a *branch* of $\mathcal{T}$ if $t_n \in \mathcal{T}$ for all n, $t_0 = \emptyset$ and $t_{n+1} \in t_n^+$ for all $n \geq 0$. If $s, t \in \mathfrak{B}$ are nodes such that neither $s \preceq t$ nor $t \preceq s$, then s and t are *incomparable*. An *antichain* in a tree is a collection of elements which are pairwise incomparable. We consider the infinite binary tree, $\mathfrak{B} = \bigcup_{n=0}^{\infty} \{0,1\}^n$, that is, finite sequences of zeros and ones. The order $\preceq$ on $\mathfrak{B}$ is defined as follows: If $s = \{s_1, s_2, \dots, s_k\} \in \{0,1\}^k \subset \mathfrak{B}$ and $t = \{t_1, t_2, \dots, t_l\} \in \{0,1\}^l \subset \mathfrak{B}$, then $s \preceq t$ if and only if $k \leq l$ and $s_i = t_i$, $1 \leq i \leq k$. As usual we denote with $|s|$ the cardinality of s, i.e. $|s| = k$. The *concatenation* of s and t is $s^\frown t = \{s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_l\} \in \{0,1\}^{k+l} \subset \mathfrak{B}$. Clearly $s \preceq s^\frown t$ and $s^+ = \{s^\frown 0, s^\frown 1\}$. The infinite binary tree is rooted with $\emptyset = \{0,1\}^0$.

Following Talagrand [Tal79, Tal84] we say that $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is an *adequate* family if

- $\mathcal{A}$ contains the empty set and the singletons: $\{n\} \in \mathcal{A}$ for all $n \in \mathbb{N}$.
- $\mathcal{A}$ is hereditary: If $A \in \mathcal{A}$ and $B \subseteq A$, then $B \in \mathcal{A}$.
- $\mathcal{A}$ is compact with respect to the topology of pointwise convergence: Given $A \subset \mathbb{N}$, if every finite subset of A is in $\mathcal{A}$, then $A \in \mathcal{A}$.

Given an adequate family $\mathcal{A}$, we define the Banach lattice $\ell_{\mathcal{A},1}$ as the set of all sequences $x = (a_i)_{i=1}^{\infty}$ satisfying $\|x\| = \sup_{A\in\mathcal{A}} \sum_{i\in A} |a_i| < \infty$ (see e.g. [AM93, Definition 2.1]). It is easy to see that, in general, the standard unit vectors $(e_i)_{i\in\mathbb{N}}$ form a normalized 1-unconditional basic sequence in $\ell_{\mathcal{A},1}$. We denote $h_{\mathcal{A},1}$ the closed subspace of $\ell_{\mathcal{A},1}$ generated by $(e_i)_{i\in\mathbb{N}}$. For example if $\mathcal{A} = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\}$, then $\ell_{\mathcal{A},1} = \ell_\infty$, $h_{\mathcal{A},1} = c_0$, and if $\mathbb{N} \in \mathcal{A}$, then $\ell_{\mathcal{A},1} = h_{\mathcal{A},1} = \ell_1$. Since $\mathcal{A}$ is compact we get that for every $x \in h_{\mathcal{A},1}$ there exists $A \in \mathcal{A}$ such that $\|P_A x\| = \|x\|$.

There is a bijection between $\mathfrak{B}$ and $\mathbb{N}$ where the natural order on $\mathbb{N}$ corresponds to the lexicographical order on $\mathfrak{B}$ (see [AT04, p. 69]). The family $\mathcal{A}$ of all subsets of $\mathbb{N}$ corresponding to the branches of $\mathfrak{B}$ and their subsets is an adequate family. We get that $X_{\mathfrak{B}} := h_{\mathcal{A},1}$ is a Banach space with 1-unconditional basis $(e_t)_{t\in\mathfrak{B}}$. It is worth pointing out that we use $t \in \mathfrak{B}$ as indices for the basis. Thus, for $x \in X_{\mathfrak{B}}$ and any non-negative integer n we write $\sum_{|t|>n} e_t^*(x)e_t$, when referring to the sum $\sum_{t\in\mathfrak{B},|t|>n} e_t^*(x)e_t$, that is, $t \in \mathfrak{B}$ is implicit. A similar notation will be used in Section 4.4.

Note that the span of the basis vectors corresponding to any infinite antichain in $X_{\mathfrak{B}}$ is isometric to c_0, and that the span of the basis vectors corresponding to any branch in $X_{\mathfrak{B}}$ is isometric to ℓ_1.

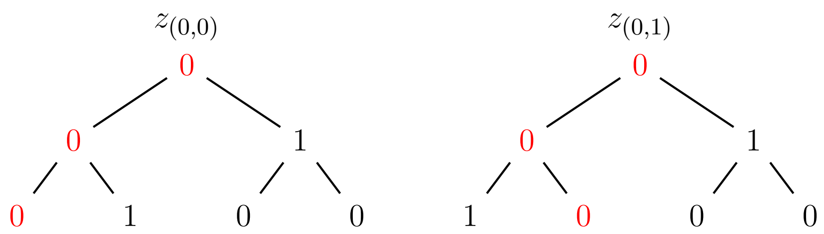
Proof of Theorem 4.3.1 (i). Consider

$$x = \sum_{|t|>0} 2^{-|t|} e_t.$$

Summing over branches we find that $\|x\| = 1$. We will show that x is a delta-point. Define $z_\emptyset = 0$ and then for $t_0 \in \mathfrak{B}$

$$z_{t_0^\frown 0} = z_{t_0} + e_{t_0^\frown 1} \quad \text{and} \quad z_{t_0^\frown 1} = z_{t_0} + e_{t_0^\frown 0}.$$

Here is a picture of $z_{(0,0)}$ and $z_{(0,1)}$:



From the definition it is clear that

$$\frac{1}{2}\left(z_{t_0^\frown 0} + z_{t_0^\frown 1}\right) = z_{t_0} + \frac{1}{2}\left(e_{t_0^\frown 0} + e_{t_0^\frown 1}\right)$$

so by induction

$$y_N := \frac{1}{2^N} \sum_{|t|=N} z_t = x - \sum_{|t|>N} 2^{-|t|} e_t.$$

Let $x^* \in S_{X^*_{\mathfrak{B}}}$ and $\delta > 0$ such that $x \in S(x^*, \delta)$. Find N such that $x^*(y_N) > 1 - \delta$ which is possible since $\| \sum_{|t|>N} 2^{-|t|} e_t \| \to 0$ as $N \to \infty$. But $x^*(y_N) > 1-\delta$ means that there exists t_0 with $|t_0| = N$ such that $x^*(z_{t_0}) > 1 - \delta$. Let $E = (t_i)_{i=1}^{\infty}$ be an infinite antichain of successors of t_0. Then $x^*(e_{t_i}) \to 0$ as $i \to \infty$. Find t_n such that

$$x^*(z_{t_0} - e_{t_n}) > 1 - \delta.$$

By definition of z_{t_0} we have $\{u \in \mathfrak{B} : u \preceq t_0\} \cap \operatorname{supp}(z_{t_0}) = \emptyset$ hence $z_{t_0} - e_{t_n} \in S(x^*, \delta)$. Summing over a branch containing t_n we get

$$\|x - (z_{t_0} - e_{t_n})\| \geq \sum_{h=1, h \neq |t_n|}^{\infty} 2^{-h} + 2^{-|t_n|} + 1 = 2$$

as desired. □

Next is the proof that $X_{\mathfrak{B}}$ does not have Daugavet-points. We first need a general lemma about Daugavet-points.

Let $(e_i)_{i\in\mathbb{N}}$ be a 1-unconditional basis in a Banach spaces X. Define

$$E_X = \{E \subset \mathbb{N} : \sum_{i\in E} e_i \in S_X\}.$$

Lemma 4.3.2. *Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $x \in S_X$ is a Daugavet-point, then $\|x - P_E x\| = 1$ for all $E \in E_X$.*

Proof. Assume $x \in S_X \cap K_X$ and that there exists $\eta > 0$ and $E \in E_X$ such that $\|x - P_E x\| < 1 - \eta$.

Define $x^* = \frac{1}{|E|} \sum_{i\in E} e_i^* \in S_{X^*}$. Choose $\gamma > 0$ such that $\max_{i\in E} \frac{e_i^*(x)}{2} < 1 - \gamma$. If $y \in S(x^*, \frac{\gamma}{|E|})$, then it follows that $1 - \gamma < e_i^*(y)$ for all $i \in E$ and

$$\|x - y\| \leq \left\| x - P_E \frac{x}{2} \right\| + \left\| y - P_E \frac{x}{2} \right\| < 2 - \frac{\eta}{2},$$

so x is not a Daugavet-point. □

Proof of Theorem 4.3.1 (ii). Assume $x \in S_{X_{\mathfrak{B}}} \cap K_{X_{\mathfrak{B}}}$. Let $E = \bigcup_{A\in M(x)} A(1)$. From Lemma 4.2.8 we see that $|E|$ is finite. Note that E is an antichain. Indeed, assume $t_0, t_1 \in E$ with $t_0 \preceq t_1$ where $A(1) = \{t_1\}$ for some $A \in M(x)$. Then since $x \in K_{X_{\mathfrak{B}}}$ and

$$1 \geq \|P_{A\cup\{t_0\}}x\| \geq \sum_{t\in A\cup\{t_0\}} e_t^*(x) = \sum_{t\in A\setminus\{t_0\}} e_t^*(x) + e_{t_0}^*(x) \geq \|P_A x\| = 1$$

we must have $t_0 = t_1$.

We have $\|x - P_E x\| < 1$ by Lemma 4.2.11 (i). From Lemma 4.3.2 we get that x is not a Daugavet-point since $E \in E_{X_{\mathfrak{B}}}$. □

Let us end this section with a remark about the proof of Theorem 4.3.1 (i). In order to prove that $X_{\mathfrak{B}}$ has a delta-point we could have used dyadic trees. Recall that a *dyadic tree* in a Banach space is a sequence $(x_t)_{t\in\mathfrak{B}}$, such that $x_t = \frac{1}{2}(x_{t^\frown 0}+x_{t^\frown 1})$.

In fact, $x = \sum_{|t|>0} 2^{-|t|}e_t$ is the root of a dyadic tree. In order to show this one uses the same z_t's as in the above proof, but attach a copy of x to the node t. Finally, we have the following result about dyadic trees and delta-points.

Proposition 4.3.3. *If a Banach space X contains a dyadic tree $(x_t)_{t\in\mathfrak{B}} \subset B_X$ such that*

$$\limsup_{n\to\infty}(\min_{|t|=n}\{\|x_\emptyset - x_t\|\}) = 2,$$

then $x_\emptyset$ is a delta-point.

Proof. Let $\varepsilon > 0$ and find n with $\|x_\emptyset - x_t\| \geq 2-\varepsilon$ for all t with $|t| = n$. This means that $x_t \in \Delta_\varepsilon(x_\emptyset)$. By definition of a dyadic tree

$$x_\emptyset = \frac{1}{2^n}\sum_{|t|=n} x_t,$$

so we have $x_\emptyset \in \operatorname{conv}\Delta_\varepsilon(x_\emptyset)$. □

4.4 A space with 1-unconditional basis and Daugavet-points

In this section we will cut of the root of the binary tree and modify the norm from the example in the previous section to allow the space to have Daugavet-points.

Let $\mathfrak{M} = \bigcup_{n=1}^\infty\{0,1\}^n$ be the binary tree with the root removed. Note that a branch $B = \{t_n\}_{n=1}^\infty$ in $\mathfrak{M}$ corresponds to the branch $\{t_n\}_{n=0}^\infty$ in $\mathfrak{B}$ where $t_0 = \emptyset$.

A λ-*segment* in $\mathfrak{M}$ is a set $S \subset \mathfrak{M}$ of the form $S = [s,t] \cup t^+$, where $[s,t]$ is a (possibly empty) segment of $\mathfrak{M}$. If $[s,t] = \emptyset$, then $S = \{(0),(1)\}$.

Using the lexicographical order $\leq$ on $\mathfrak{M}$ we have a bijective correspondence to $\mathbb{N}$ with the natural order. Let $\mathcal{A}$ be the adequate family of subsets of $\mathbb{N}$ corresponding to subsets of branches and subsets of λ-segments. Using this adequate family we get a Banach space $X_{\mathfrak{M}} := h_{\mathcal{A},1}$ with 1-unconditional basis $(e_t)_{t\in\mathfrak{M}}$. We call $X_{\mathfrak{M}}$ the modified binary tree space. Note that $X_{\mathfrak{M}}$ contains isometric copies of c_0 and ℓ_1 just like $X_{\mathfrak{B}}$.

As we saw in the proof of Theorem 4.3.1 (ii) the antichains in the tree play an important role for the existence of Daugavet-points.

Define

$$\mathfrak{F} := \{0\} \cup \{z \in S_{X_{\mathfrak{M}}} : z(\mathfrak{M}) \subset \{0, \pm 1\}\}.$$

The set $E_{X_{\mathfrak{M}}}$ from Section 4.3 can be described as the set of all non-void finite antichains E of $\mathfrak{M}$ such that $|A \cap E| \leq 1$ for all $A \in \mathcal{A}$. Clearly supp$(z) \in E_{X_{\mathfrak{M}}}$ for every $z \in \mathfrak{F} \setminus \{0\}$ and every z with supp$(z) \in E_{X_{\mathfrak{M}}}$ and $z(\mathfrak{M}) \subset \{0, \pm 1\}$ belongs to $\mathfrak{F}$. It is also clear that for every $E \in E_{X_{\mathfrak{M}}}$ there exists a branch B such that $B \cap E = \emptyset$. We will see in Lemma 4.4.1 and Theorem 4.4.2 that the sets $E_{X_{\mathfrak{M}}}$ and $\mathfrak{F}$ will play an essential role in characterizing the Daugavet-points of $X_{\mathfrak{M}}$.

If M is a finite subset of $\mathfrak{M}$, then we will use the notation $K_M = \{\sum_{t\in M} a_t e_t : a_t \geq 0\}$ and $\mathfrak{F}_M = \{z \in \mathfrak{F} : \text{supp}(z) \subset M\}$.

First we prove a lemma which says that convex combinations of elements in $\mathfrak{F}$ are dense in the unit ball of $X_{\mathfrak{M}}$.

Lemma 4.4.1. *Let M be a finite subset of $\mathfrak{M}$. Then*

$$\text{span}\,\{e_t : t \in M\} \cap B_{X_{\mathfrak{M}}} = \text{conv}\,(\mathfrak{F}_M)$$

that is, for every $x \in$ span $\{e_t : t \in M\} \cap B_{X_{\mathfrak{M}}}$ we have

$$x = \sum_{k=1}^{N} \lambda_k z_k \tag{4.1}$$

where $z_k \in \mathfrak{F}_M$, $\lambda_k > 0$, $\sum_{k=1}^{N} \lambda_k = 1$. In particular, ext$(K_M \cap B_{X_{\mathfrak{M}}}) = K_M \cap \mathfrak{F}_M$.

Proof. With M_n denote the subset of $\mathfrak{M}$ which corresponds to $\{1, \ldots, n\} \subset \mathbb{N}$. We will show, by induction, that for every $x \in K_{M_{2n}} \cap B_{X_{\mathfrak{M}}}$ we have

$$x = \sum_{k=1}^{N} \lambda_k z_k,$$

where $z_k \in K_{\text{supp}(x)} \cap \mathfrak{F}$, $\lambda_k > 0$ and $\sum_{k=1}^N \lambda_k = 1$. As $K_M \subseteq K_{M_{2n}}$ for some $n \in \mathbb{N}$ and $z_k \in K_{\text{supp}(x)} \cap \mathfrak{F}$, the result will follow.

The base step is $x \in K_{M_2} \cap B_{X_{\mathfrak{M}}}$ with $e_t^*(x) \geq 0$ for $t \in M_2 = \{(0),(1)\}$. Write $e_{(0)}^*(x) = a_0$ and $e_{(1)}^*(x) = a_1$. Define $c = 1 - a_0 - a_1$, $z_0 = e_{(0)}$, and $z_1 = e_{(1)}$. Then

$$x = (c \cdot 0 + a_0 z_0 + a_1 z_1)$$

is a convex combination of elements in $K_{\text{supp}(x)} \cap \mathfrak{F}$.

Assume the induction hypothesis holds for $n \in \mathbb{N}$. Let $x \in K_{M_{2(n+1)}} \cap B_{X_{\mathfrak{M}}}$. Let $t \in \mathfrak{M}$ be the node such that $t^\frown 0$ corresponds to $2n+1$ and $t^\frown 1$ to $2n+2$. Define

$$x' = x - e_{t^\frown 0}^*(x) e_{t^\frown 0} - e_{t^\frown 1}^*(x) e_{t^\frown 1}.$$

By assumption we have $x' = \sum_{k=1}^N \lambda_k z_k$ with $\lambda_k > 0$, $\sum_{k=1}^N \lambda_k = 1$ and $z_k \in K_{\text{supp}(x')} \cap \mathfrak{F}$.

Define the segment $A = \{s \in \mathfrak{M} : s \preceq t\}$ and the sets

$$I = \{k \in \{1,\dots,N\} : P_A z_k = 0\} \quad \text{and} \quad J = \{1,\dots,N\} \setminus I.$$

For $k \in I$ we let

$$z_{k,0} := z_k + e_{t^\frown 0} \quad \text{and} \quad z_{k,1} := z_k + e_{t^\frown 1}.$$

Since $z_k \in K_{\text{supp}(x')} \cap \mathfrak{F}$ we get $z_{k,0}$, $z_{k,1} \in K_{\text{supp}(x)} \cap \mathfrak{F}$ and

$$\sum_{s \in A} e_s^*(x') = \sum_{s \in A} e_s^*(x) = \sum_{k \in J} \lambda_k.$$

Thus, by definition of the norm we have,

$$0 \leq e_{t^\frown 0}^*(x) + e_{t^\frown 1}^*(x) \leq 1 - \sum_{s \in A} e_s^*(x) = \sum_{k \in I} \lambda_k.$$

Write $e_{t^\frown 0}^*(x) = a_0$ and $e_{t^\frown 1}^*(x) = a_1$. Define $c = \sum_{k \in I} \lambda_k - a_0 - a_1$. Let $m = \sum_{k \in I} \lambda_k$. It follows that

$$\begin{aligned}
x &= x' + a_0 e_{t^\frown 0} + a_1 e_{t^\frown 1} \\
&= \sum_{k \in J} \lambda_k z_k + \sum_{k \in I} \lambda_k z_k + \sum_{k \in I} \lambda_k \left(\frac{a_0}{m} e_{t^\frown 0} + \frac{a_1}{m} e_{t^\frown 1} \right) \\
&= \sum_{k \in J} \lambda_k z_k + \sum_{k \in I} \lambda_k \frac{(a_0 + a_1 + c)}{m} z_k + \sum_{k \in I} \lambda_k \left(\frac{a_0}{m} e_{t^\frown 0} + \frac{a_1}{m} e_{t^\frown 1} \right) \\
&= \sum_{k \in J} \lambda_k z_k + \sum_{k \in I} \lambda_k \left(\frac{a_0}{m} z_{k,0} + \frac{a_1}{m} z_{k,1} + \frac{c}{m} z_k \right)
\end{aligned}$$

which is a convex combination of elements in $K_{\text{supp}(x)} \cap \mathfrak{F}$. $\square$

With the above lemma in hand we are able to characterize Daugavet-points in $X_{\mathfrak{M}}$ in terms of $E_{X_{\mathfrak{M}}}$. This will give us an easy way to identify and give examples of Daugavet-points.

Theorem 4.4.2. *Let $x \in S_{X_{\mathfrak{M}}}$, then the following are equivalent*

(i) x is a Daugavet-point;

(ii) $\|x - P_E x\| = 1$, for all $E \in E_{X_{\mathfrak{M}}}$;

(iii) for any $z \in \mathfrak{F}$, either $\|x - z\| = 2$ or for all $\varepsilon > 0$ there exists $s \in \mathfrak{M}$ such that $z \pm e_s \in \mathfrak{F}$ and $\|x - z \pm e_s\| > 2 - \varepsilon$.

Proof. As usual we will assume that $x \in K_{X_{\mathfrak{M}}}$ throughout.

(i) $\Rightarrow$ (ii) is Lemma 4.3.2.

(ii) $\Rightarrow$ (iii). Let $\varepsilon > 0$, $z \in \mathfrak{F}$ and $E = \text{supp}(z)$. We have assumed that $\|x - P_E x\| = 1$.

By definition of $M(x - P_E x)$ we have $A \cap E = \emptyset$ for every $A \in M(x - P_E x)$. If there, for some $A \in M(x - P_E x)$, exists $t \in E$ and $s_0 \in A$ such that $t \preceq s_0$, or $t \in E$ such that $s \preceq t$ for all $s \in A$, then we are done since $e_t^*(x) = 0$ and

$$\|x - z\| \geq \sum_{s \in A} |e_s^*(x)| + |e_t^*(z)| = 2.$$

So from now on we assume that no such A exists.

Assume that there exists $A \in M(x - P_E x)$ that is a subset of a branch B. By definition of the norm, we have $e_t^*(x) = 0$ for $t \in B \setminus A$, and by the assumption above, we also have $B \cap E = \emptyset$. Since $|e_t^*(x)| \to 0$ as $|t| \to \infty$ for $t \in B$ we can find $s \in B$ with $|e_s^*(x)| < \varepsilon/2$ and hence

$$\|x - z \pm e_s\| \geq \sum_{t \in A, t \neq s} |e_t^*(x)| + |e_s^*(x) \pm 1| \geq 2 - \varepsilon.$$

This concludes the case where A is a subset of a branch.

Suppose for contradiction that no $A \in M(x - P_E x)$ is a subset of a branch, then every $B \in M(x - P_E x)$ is a subset of a λ-segment. By Lemma 4.2.10 we must have $|M(x - P_E x)| < \infty$.

Choose any $B \in M(x - P_E x)$ and write

$$B = \{b_1 \prec b_2 \prec \cdots \prec b_n\} \cup \{b^\frown 0, b^\frown 1\},$$

where $b_n \preceq b$. In particular $e_s^*(x) \neq 0$ for $s \in b^+$.

Let $R = \{t \in E : b^\frown 0 \prec t\}$ and $E_1 = (E \cup \{b^\frown 0\}) \setminus R$. From the assumptions above $E \cap \{t : t \preceq b^\frown 0\} = \emptyset$, so $E_1 \in E_{X_{\mathfrak{M}}}$.

Let $C \in M(x - P_{E_1}x)$. Notice that $C \cap \{t : b^\frown 0 \preceq t\} = \emptyset$. Otherwise, by definition of the norm, we get the contradiction

$$1 = \|P_{C\cap\{b^\frown 0\}}x\| = \sum_{t\in C} |e_t^*(x)| + |e_{b^\frown 0}^*(x)| > \sum_{t\in C} |e_t^*(x)| = \|P_C x\| = 1.$$

Hence $P_C(x - P_{E_1}x) = P_C(x - P_E x)$ and $C \in M(x - P_E x)$.

We have $M(x - P_{E_1}x) \subseteq M(x - P_E x)$, but since $B \cap E_1 \neq \emptyset$ we have $B \notin M(x - P_{E_1}x)$ so the inclusion is strict.

We now have $|M(x - P_{E_1}x)| < |M(x - P_E x)|$ and no $C \in M(x - P_{E_1}x)$ is a subset of a branch. We can use the argument above a finite number of times until we are left with $E_m \in E_{X_{\mathfrak{M}}}$ with $\|x - P_{E_m}x\| = 1$ and $M(x - P_{E_m}x) = \emptyset$ which contradicts Lemma 4.2.7.

Finally, (iii) $\Rightarrow$ (i). Choose $\varepsilon > 0$. Let $y \in B_{X_{\mathfrak{M}}}$ with finite support. Then by Lemma 4.4.1, we can write $y = \sum_{k=1}^n \lambda_k z_k$, with $z_k \in \mathfrak{F}$, $\lambda_k \geq 0$ and $\sum_{k=1}^n \lambda_k = 1$. Let $D_1 = \{k \in \{1, \ldots, n\} : \|x - z_k\| = 2\}$ and $D_2 = \{1, \ldots, n\} \setminus D_1$. We can, by assumption, for each $k \in D_2$ find $s_k \in \mathfrak{M}$ such that $z_k \pm e_{s_k} \in \mathfrak{F}$ with $\|x - z_k \pm e_{s_k}\| > 2 - \varepsilon$. Then $y \in \operatorname{conv} \Delta_\varepsilon(x)$ since

$$y = \sum_{i\in D_1} \lambda_k z_k + \sum_{k\in D_2} \frac{\lambda_k}{2}(z_k + e_{s_k}) + \sum_{k\in D_2} \frac{\lambda_k}{2}(z_k - e_{s_k}).$$

The set of all such y is dense in $B_{X_{\mathfrak{M}}}$, hence $B_{X_{\mathfrak{M}}} = \overline{\operatorname{conv}}\Delta_\varepsilon(x)$ so x is a Daugavet-point. ☐

Corollary 4.4.3. *If $x \in S_{X_{\mathfrak{M}}}$ such that $\|P_A x\| = 1$ for all branches A, then x is a Daugavet-point.*

Proof. Let $E \in E_{X_{\mathfrak{M}}}$. There exists a branch B such that $B \cap E = \emptyset$. Then $\|x - P_E x\| \geq \|P_B x\| = 1$. By Theorem 4.4.2 x is a Daugavet-point. ☐

With a characterization of Daugavet-points in hand we can now prove the main result of this section.

Theorem 4.4.4. *In $X_{\mathfrak{M}}$ we have that*

(i) there exists $x \in S_{X_{\mathfrak{M}}}$ which is a Daugavet-point;

(ii) there exists $w \in S_{X_{\mathfrak{M}}}$ which is a delta-point, but not a Daugavet-point.

Proof. Let $x = \sum_{t \in \mathfrak{M}} 2^{-|t|} e_t$. We have that x is a Daugavet-point by Corollary 4.4.3.

The next part of the proof is similar to the proof of Theorem 4.3.1 (i). We will show that a shifted version of x is a delta-point which is not a Daugavet-point. Define an operator on the modified binary tree:

$$L\left(\sum_{|t|>0} a_t e_t\right) = \sum_{|t|\geq 0} a_{0\frown t} e_{0\frown t} + \sum_{|t|\geq 0} a_{1\frown t} e_{(1,0)\frown t},$$

where $t = \emptyset$ when $|t| = 0$.

Define $w = L(x)$. Let $x^* \in S_{X^*_{\mathfrak{M}}}$ and $\delta > 0$ such that $w \in S(x^*, \delta)$. Just as in the proof of Theorem 4.3.1 (i) we can find $z_{t_0} \in S_{X_{\mathfrak{M}}}$ whose support is an antichain (i.e. $z_{t_0} \in \mathfrak{F}$) and we can find e_{t_n} such that $z_{t_0} - e_{t_n} \in S(x^*, \delta)$. Summing over a branch containing t_n we get $\|w - (z_{t_0} - e_{t_n})\| = 2$.

Let $E = \{(0), (1, 0)\}$. Then $\|w - P_E w\| = \sum_{i=2}^{\infty} 2^{-i} = \frac{1}{2} < 1$ so by Theorem 4.4.2 w is not a Daugavet-point. □

In [AHLP20], the property that the unit ball of a Banach space is the closed convex hull of its delta-points was studied. We will next show that $X_{\mathfrak{M}}$ satisfies something much stronger, the unit ball is the closed convex hull of a subset of its Daugavet-points.

If D is the set of all Daugavet-points in $X_{\mathfrak{M}}$ define

$$D_B = \{x \in D : \|P_B x\| = 1 \text{ for all branches } B \text{ of } \mathfrak{M}\}.$$

The proof of Theorem 4.4.4 shows that D_B is non-empty.

For $t_0 \in \mathfrak{M}$, let S_{t_0} be the shift operator on $X_{\mathfrak{M}}$ that shifts the root to t_0, that is

$$S_{t_0}\left(\sum_{t \in \mathfrak{M}} a_t e_t\right) = \sum_{t \in \mathfrak{M}} a_t e_{t_0^\frown t} \tag{4.2}$$

It is clear that S_{t_0} is an isometry on $X_{\mathfrak{M}}$.

Proposition 4.4.5. *The space $X_{\mathfrak{M}}$ satisfies $B_{X_{\mathfrak{M}}} = \overline{conv}\,(D_B)$.*

Proof. Let $y \in B_{X_{\mathfrak{M}}}$. We may assume that y has finite support, since such y are dense in $B_{X_{\mathfrak{M}}}$. By Lemma 4.4.1, we can write $y = \sum_{k=1}^{n} \lambda_k z_k$ where $z_k \in \mathfrak{F}$, $\lambda_k \geq 0$ and $\sum_{k=1}^{n} \lambda_k = 1$.

Fix $z \in \mathfrak{F}$. Let $m := \max\{|t| : t \in \operatorname{supp}(z)\}$.

$$\mathcal{B} = \{t \in \mathfrak{M} : |t| = m, \sum_{s \preceq t} |e_s^*(z)| = 0\}.$$

Choose any $x_0 \in D_B$ and use the shift operator in (4.2) to define

$$x := \sum_{t \in \mathcal{B}} S_t(x_0).$$

Observe that $z \pm x$ takes its norm along every branch, so by Corollary 4.4.3 both $z \pm x \in D_B$.

Repeat this construction for z_k to create x_k for $k \in \{1, \ldots, n\}$. Then

$$y = \sum_{k=1}^{n} \frac{\lambda_k}{2}(z_k + x_k) + \sum_{k=1}^{n} \frac{\lambda_k}{2}(z_k - x_k),$$

is a convex combination of Daugavet-points in D_B. □

Our next result is that $X_{\mathfrak{M}}$ has the remarkable property that the Daugavet-points are weakly dense in the unit ball. So in a sense there are lots of Daugavet-points, but of course not enough of them in order for $X_{\mathfrak{M}}$ to have the Daugavet property. First we need a lemma. For $t \in \mathfrak{M}$, S_t denotes the shift operator defined in (4.2) above.

Lemma 4.4.6. *Let $x^* \in S_{X_{\mathfrak{M}}^*}$ and $s \in \mathfrak{B}$. For any $x \in S_{X_{\mathfrak{M}}}$ and $\varepsilon > 0$ there exist some infinite antichain $E = \{t_i\}_{i=1}^{\infty}$ with the following properties*

(i) $\left\|\sum_{i=1}^{n} e_{t_i}\right\| = 1$ *for all* $n \in \mathbb{N}$;

(ii) $s \preceq t$ *for all* $t \in E$;

(iii) $|x^*(S_t x)| < \varepsilon$ *for all* $t \in E$.

Proof. Pick any $x^* \in S_{X_{\mathfrak{M}}^*}$, $s \in \mathfrak{B}$ and $x \in S_{X_{\mathfrak{M}}}$. It is not difficult to find an infinite antichain $E = \{t_i\}_{i=1}^{\infty}$ satisfying (i) and (ii). Since E is an antichain we have $\|\sum_{i=1}^{n} S_{t_i}(x)\| = 1$ for all $n \in \mathbb{N}$. Hence

$$\lim_{i \to \infty} x^*\left(S_{t_i} x\right) = 0,$$

and then we can find $n \in \mathbb{N}$ such that $|x^*\left(S_{t_i} x\right)| < \varepsilon$ for all $i \geq n$. Now $E' = E \setminus \{t_i\}_{i=1}^{n}$ satisfies (i), (ii) and (iii). □

Theorem 4.4.7. *In $X_{\mathfrak{M}}$ every non-empty relatively weakly open subset of $B_{X_{\mathfrak{M}}}$ contains a Daugavet-point.*

Proof. Since vectors with finite support are norm dense in $B_{X_{\mathfrak{M}}}$, it enough show that for any $y \in B_{X_{\mathfrak{M}}}$ with finite support and any relatively weakly open neighbourhood of y of the form

$$W := \{x \in B_{X_{\mathfrak{M}}} : |x_i^*(y - x)| < \varepsilon, i = 1, \dots, n\},$$

where $x_i^* \in S_{X_{\mathfrak{M}}^*}, i = 1, \dots, n$ and $\varepsilon > 0$, contains a Daugavet-point.

Let $m := \max\{|t| : t \in \operatorname{supp}(y)\}$, and for $t \in \mathfrak{M}$ with $|t| = m$ define

$$\mu_t := 1 - \sum_{s \preceq t} |e_s^*(y)|$$

and

$$\mathcal{N} := \{t \in \mathfrak{M} : |t| = m, \mu_t > 0\}.$$

From Corollary 4.4.3 we have that $g = \sum_{s \in \mathfrak{M}} 2^{-|s|} e_s$ is a Daugavet-point. By Lemma 4.4.6 for each $t \in \mathcal{N}$ there exists $t \preceq b_t$ such that $|x_i^*(S_{b_t} g)| < \varepsilon / 2^m$ for $i = 1, \dots, n$. Now put

$$x = y + \sum_{t \in \mathcal{N}} \mu_t S_{b_t}(g).$$

By construction $x \in S_{X_{\mathfrak{M}}}$ and we have $x \in W$ since

$$|x_i^*(y - x)| = \left| x_i^* \left(\sum_{t \in \mathcal{N}} \mu_t S_{b_t}(g) \right) \right| \leq \sum_{t \in \mathcal{N}} \mu_t |x_i^*(S_{b_k} g)| < \frac{\varepsilon}{2^m} \sum_{t \in \mathcal{N}} \mu_t < \varepsilon.$$

Using Theorem 4.4.2 we will show that x is a Daugavet-point. Indeed, let $E \in E_{X_{\mathfrak{M}}}$. Then there exists a branch A with $A \cap E = \emptyset$. Let $t \in A$ with $|t| = m$. If $t \notin \mathcal{N}$, then

$$\|x - P_E x\| \geq \sum_{s \preceq t} |e_s^*(y)| = 1.$$

If $t \in \mathcal{N}$, then since $S_{b_t}(g)$ is a Daugavet-point, there exists a branch B with $t \in B$ such that $\|S_{b_t}(g) - P_E S_{b_t}(g)\| = \sum_{s \in B} |S_{b_t}(g)_s| = 1$. Thus

$$\|x - P_E x\| \geq \sum_{s \preceq t} |e_s^*(y)| + \sum_{\substack{s \in B, \\ s \succ b_t}} \mu_t |S_{b_t}(g)_s| = 1 - \mu_t + \mu_t = 1,$$

and we are done. $\square$

Question 2. How "massive" does the set of Daugavet-points in S_X have to be in order to ensure that a Banach space X fails to have an unconditional basis?

If S is a slice of the unit ball of $X_{\mathfrak{M}}$, then the above proposition tells us that S contains a Daugavet-point x. Then by definition of Daugavet-points there exists for any $\varepsilon > 0$ a $y \in S$ with $\|x - y\| \geq 2 - \varepsilon$. Thus the diameter of every slice of the unit ball of $X_{\mathfrak{M}}$ is 2, that is $X_{\mathfrak{M}}$ has the *local diameter two property*.

The next natural question is whether the diameter of every non-empty relatively weakly open neighborhood in $B_{X_{\mathfrak{M}}}$ equals 2, that is, does $X_{\mathfrak{M}}$ have the *diameter two property*? The answer is no, in fact, every Daugavet-point in D_B has a weak neighborhood of arbitrary small diameter. Let us remark that the first example of a Banach space with the local diameter two property, but failing the diameter two property was given in [BGLPRZ15]. While we have used binary trees, their construction used the tree of finite sequences of positive integers and they even showed that every Banach space containing c_0 can be renormed to have the local diameter two property and fail the diameter two property.

Proposition 4.4.8. *In $X_{\mathfrak{M}}$ every $x \in D_B$ is a point of weak- to norm-continuity for the identity map on $B_{X_{\mathfrak{M}}}$. In particular, $X_{\mathfrak{M}}$ fails the diameter two property.*

Proof. Let $\varepsilon > 0$ and $x \in D_B$. Let $n \in \mathbb{N}$ be such that $\|\sum_{|t|>n} x_t e_t\| < \frac{\varepsilon}{8}$. Consider the weak neighborhood W of x

$$W = \{y \in B_{X_{\mathfrak{M}}} : |e_t^*(x-y)| < \frac{\varepsilon}{2^{|t|+3}}, |t| \leq n\}.$$

We want to show that the diameter of W is less than ε. Let $y = \sum_{t \in \mathfrak{M}} y_t e_t \in W$. Let A be a subset of a branch or of a λ-segment in $\mathfrak{M}$. Since $|x_t - y_t| < \varepsilon 2^{-|t|-3}$ for $|t| \leq n$, $\|\sum_{|t|>n} x_t e_t\| < \frac{\varepsilon}{8}$, and x attains its norm along every branch of $\mathfrak{M}$, we have

$$\sum_{\substack{t \in A \\ |t| \leq n}} |y_t| > \sum_{\substack{t \in A \\ |t| \leq n}} |x_t| - |x_t - y_t| > \sum_{\substack{t \in A \\ |t| \leq n}} |x_t| - \frac{\varepsilon}{8} > 1 - \frac{\varepsilon}{4}.$$

Hence $\sum_{\substack{t \in A \\ |t|>n}} |y_t| < \frac{\varepsilon}{4}$, and thus

$$\begin{aligned}
\sum_{t \in A} |x_t - y_t| &= \sum_{\substack{t \in A \\ |t| \leq n}} |x_t - y_t| + \sum_{\substack{t \in A \\ |t|>n}} |x_t - y_t| \\
&< \sum_{\substack{t \in A \\ |t| \leq n}} \varepsilon 2^{-|t|-3} + \sum_{\substack{t \in A \\ |t|>n}} |x_t| + \sum_{\substack{t \in A \\ |t|>n}} |y_t| \\
&< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

From this it follows that the diameter of W is less than ε. $\square$

Recall from [ALL16] that a Banach space X is *locally almost square* if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$.

It is known that every locally almost square Banach space X has the local diameter two property. As noted above $X_{\mathfrak{M}}$ has the local diameter two property, but it is not locally almost square as the following proposition shows.

Proposition 4.4.9. *$X_{\mathfrak{M}}$ is not locally almost square.*

Proof. Consider $x = \frac{1}{4}e_{(0)} + \frac{3}{4}e_{(1)}$. Let $0 < \varepsilon < \frac{1}{4}$ and suppose there exists $y = \sum_{t \in \mathfrak{M}} y_t e_t \in S_{X_{\mathfrak{M}}}$ with $\|x \pm y\| \leq 1 + \varepsilon < \frac{5}{4}$. Then clearly $|y_{(1)}| \leq \frac{1}{4} + \varepsilon$. By considering $-y$ if necessary we may assume that $y_{(1)} \geq 0$. Then

$$\begin{aligned} 1 + \varepsilon &\geq \max_{\pm}\{|\frac{1}{4} \pm y_{(0)}| + |\frac{3}{4} \pm y_{(1)}|\} \\ &\geq |\frac{1}{4} - y_{(0)}| + \frac{3}{4} + |y_{(1)}| \\ &\geq |y_{(0)}| - \frac{1}{4} + \frac{3}{4} + |y_{(1)}|, \end{aligned}$$

which yields $|y_{(0)}| + |y_{(1)}| \leq \frac{1}{2} + \varepsilon < \frac{3}{4}$. Thus since $\|y\| = 1$ there must exist a subset A of a branch or a λ-segment such that $|A \cap \{(0), (1)\}| = 1$ and $\sum_{t \in A} |y_t| = 1$. Let $s \in A \cap \{(0), (1)\}$.

$$\frac{5}{4} > \|x \pm y\| = \max_{\pm} |x_s \pm y_s| + \sum_{\substack{t \in A \\ t \neq s}} |y_t| = |x_s| + |y_s| + 1 - |y_s|$$

and we get the contradiction $|x_s| < \frac{1}{4}$. $\square$

Recall from [HLP15] that a Banach space X is *locally octahedral* if for every $x \in S_X$ and $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x \pm y\| \geq 2 - \varepsilon$.

It is known that every Banach space with the Daugavet property is octahedral. Even though the modified binary tree space have lots of Daugavet-points, as seen in Proposition 4.4.5, it is not even locally octahedral.

Proposition 4.4.10. *$X_{\mathfrak{M}}$ is not locally octahedral.*

Proof. Consider $x = \frac{1}{2}(e_{(0)} + e_{(1)}) \in S_{X_{\mathfrak{M}}}$. We want to show that for all $y \in S_{X_{\mathfrak{M}}}$ we have $\min \|x \pm y\| \leq \frac{3}{2}$.

Let $y = \sum_{t\in\mathfrak{M}} y_t e_t \in S_{X_\mathfrak{M}}$. Let A be a subset of a branch or a λ-segment. If $A \neq \{(0),(1)\}$, then

$$\sum_{t\in A} |x_t \pm y_t| \leq \begin{cases} \frac{1}{2} + \sum_{t\in A} |y_t|; & A\cap\{(0),(1)\} \neq \emptyset \\ \sum_{t\in A} |y_t|; & A\cap\{(0),(1)\} = \emptyset \end{cases} \leq \begin{cases} \frac{3}{2} \\ 1 \end{cases}$$

If $A = \{(0),(1)\}$, then, since $|y_{(0)}| + |y_{(1)}| \leq 1$ and a convex function attains its maximum at the extreme points, we get

$$|\frac{1}{2} + y_{(0)}| + |\frac{1}{2} + y_{(1)}| + |\frac{1}{2} - y_{(0)}| + |\frac{1}{2} - y_{(1)}| \leq 3.$$

Hence $\min \|x \pm y\| \leq \frac{3}{2}$. □

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5 Delta-points in Banach spaces generated by adequate families

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Submitted (2021).

Abstract

We study delta-points in Banach spaces $h_{\mathcal{A},p}$ generated by adequate families $\mathcal{A}$ where $1 \leq p < \infty$. When the familiy $\mathcal{A}$ is regular and $p = 1$, these spaces are known as combinatorial Banach spaces. When $p > 1$ we prove that neither $h_{\mathcal{A},p}$ nor its dual contain delta-points. Under the extra assumption that $\mathcal{A}$ is regular, we prove that the same is true when $p = 1$. In particular the Schreier spaces and their duals fail to have delta-points. If $\mathcal{A}$ consists of finite sets only we are able to rule out the existence of delta-points in $h_{\mathcal{A},1}$ and Daugavet-points in its dual.

We also show that if $h_{\mathcal{A},1}$ is polyhedral, then it is either (I)-polyhedral or (V)-polyhedral (in the sense of Fonf and Veselý).

5.1 Introduction

According to Talagrand [Tal79, Tal84], a family $\mathcal{A}$ of subsets of $\mathbb{N}$ is *adequate* if $\mathcal{A}$ contains the singletons, is hereditary and is compact with respect to the topology of pointwise convergence. We study the spaces $h_{\mathcal{A},p}$, $1 \leq p < \infty$, generated by an adequate family $\mathcal{A}$ by completing the space c_{00} of all finitely supported sequences with the norm

$$\|\sum_{i=1}^{n} a_i e_i\| = \sup_{A \in \mathcal{A}} (\sum_{i \in A} |a_i|^p)^{1/p}.$$

Examples of $h_{\mathcal{A},p}$ spaces are c_0, ℓ_p, $\ell_1(c_0)$ and the (higher order) Schreier space(s). We study the presence or absence of *Daugavet*- and *delta*-points in $h_{\mathcal{A},p}$ spaces and their duals.

For a Banach space X we denote by B_X, S_X and X^* the closed unit ball, the unit sphere and the topological dual of X respectively. Recall that a *slice* of B_X is a set

$$S(x^*, \delta) = \{x \in B_X \ : \ x^*(x) > 1 - \delta\},$$

where $x^* \in S_{X^*}$ and $\delta > 0$. Following [AHLP20] we say that $x \in S_X$ is a Daugavet-point (resp. delta-point) if every element in the unit ball (resp. x itself) is in the closed convex hull of unit ball elements that are almost at distance 2 from x. In [AHLP20, Lemmas 2.2 and 2.3], we find the following characterization of Daugavet- and delta-points which will serve as our definitions.

Lemma 5.1.1. *Let X be a Banach space. Then $x \in S_X$ is a*

(i) delta-point *if and only if for every slice $S(x^*, \delta)$ of B_X with $x \in S(x^*, \delta)$ and for every $\varepsilon > 0$ there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.*

(ii) Daugavet-point *if and only if for every slice $S(x^*, \delta)$ of B_X and for every $\varepsilon > 0$ there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.*

A Banach space has the Daugavet property if (and only if) every $x \in S_X$ is a Daugavet-point. For more on Daugavet- and delta-points see [AHLP20, ALMT21, HPV21, JRZ20, RZ20].

It is known that if a Banach space X has the Daugavet property, then X does not have an unconditional basis [Kad96, Corollary 2.3]. In [ALMT21, Theorem 4.7] it was shown that there exists an $h_{\mathcal{A},1}$ space (with 1-unconditional basis) with "lots" of Daugavet-points, in the sense that its unit ball is the weak closure of its Daugavet-points. After some preliminaries in Section 5.2, we show in Section 5.3 that if $1 < p < \infty$ then neither $h_{\mathcal{A},p}$ nor $h^*_{\mathcal{A},p}$ have delta-points.

The case $p = 1$ studied in Section 5.4 is not so clear cut. As mentioned above there exist $h_{\mathcal{A},1}$ spaces with delta-points, but if $\mathcal{A}$ consists of all subsets of $\mathbb{N}$, then $h_{\mathcal{A},1} = \ell_1$ and by [ALMT21, Theorem 2.17] ℓ_1 does not have delta-points. We show that if the adequate family $\mathcal{A}$ contains an infinite set then $h^*_{\mathcal{A},1}$ always has delta-points. Our main focus in Section 5.4 is on adequate families of finite sets. If $\mathcal{A}$ is an adequate family of finite subsets it follows from the results of [ALMT21] that $h_{\mathcal{A},1}$ does not have delta-points. If $\mathcal{A}$ is in addition spreading (see Section 5.2) then the $h_{\mathcal{A},1}$ spaces we get are the combinatorial Banach spaces studied by Antunes, Beanland and Chu in [ABC19]. Using results from [ABC19] we show that if $h_{\mathcal{A},1}$ is a combinatorial Banach space, then also $h^*_{\mathcal{A},1}$ fail to have delta-points.

In Section 5.4 we also show that all extreme points of $h^*_{\mathcal{A},1}$ are w^*-exposed for all families $\mathcal{A}$. If $\mathcal{A}$ consists of finite sets only, then the extreme points in $h^*_{\mathcal{A},1}$ are in fact w^*-strongly exposed. We use this observation to show that such $h^*_{\mathcal{A},1}$ spaces fail to have Daugavet-points.

Finally, in Section 5.5 we study polyhedrality in $h_{\mathcal{A},1}$ spaces. A Banach space is *polyhedral* if the unit ball of every finite-dimensional subspace of X is a polytope [Kle60]. In [ABC19] it was shown that combinatorial Banach spaces are all (V)-polyhedral (in the sense of Fonf and Veselý [FV04]). In fact, their proofs also work without the spreading property and show that all $h_{\mathcal{A},1}$ spaces where the adequate family consists of finite subsets of $\mathbb{N}$ are (V)-polyhedral. We observe that $h_{\mathcal{A},1}$ is polyhedral if and only if $h_{\mathcal{A},1}$ is (V)-polyhedral if and only if $\mathcal{A}$ is an adequate family of finite sets. Furthermore, we show that $h_{\mathcal{A},1}$ is (I)-polyhedral if and only if $h_{\mathcal{A},1}$ is (IV)-polyhedral if and only if $\{A \in \mathcal{A} : i \in A\}$ is finite for all $i \in \mathbb{N}$.

We conclude the paper with some questions arising from the present work.

5.2 Preliminaries

Recall that a Schauder basis $(e_i)_{i\in\mathbb{N}}$ of a Banach space X is a *1-unconditional basis* if for all $N \in \mathbb{N}$ and all scalars $a_1, \dots, a_N$, $b_1, \dots, b_N$ such that $|a_i| \le |b_i|$ for $i = 1, \dots, N$ then the following inequality holds:

$$\left\| \sum_{i=1}^{N} a_i e_i \right\| \le \left\| \sum_{i=1}^{N} b_i e_i \right\|.$$

A basis $(e_i)_{i\in\mathbb{N}}$ is *normalized* if $\|e_i\| = 1$ for all i and it is *shrinking* if the biorthogonal functionals $(e_i^*)_{i\in\mathbb{N}}$ is a basis for X^*. For $x \in X$ the *support* of x is defined by $\text{supp}(x) = \{i \in \mathbb{N} : e_i^*(x) \neq 0\}$.

If $(e_i)_{i\in\mathbb{N}}$ is 1-unconditional, then by the classic result of James it is shrinking if and only if X does not contain an isomorphic copy of ℓ_1. If $(e_i)_{i\in\mathbb{N}}$ is a 1-unconditional basis then for any $A \subset \mathbb{N}$ the projection P_A defined by

$$P_A(\sum_{i\in\mathbb{N}} x_i e_i) = \sum_{i\in A} x_i e_i$$

satisfies $\|P_A\| \le 1$.

As mentioned earlier we will study the existence of delta- and Daugavet-points in sequence spaces with 1-unconditional bases generated by adequate families of subsets of $\mathbb{N}$.

Definition 5.2.1. A family $\mathcal{A}$ of subsets of $\mathbb{N}$ is *adequate* if

(i) $\mathcal{A}$ contains the empty set and the singletons: $\{i\} \in \mathcal{A}$ for all $i \in \mathbb{N}$.

(ii) $\mathcal{A}$ is hereditary: If $A \in \mathcal{A}$ and $B \subseteq A$, then $B \in \mathcal{A}$.

(iii) $\mathcal{A}$ is compact with respect to the topology of pointwise convergence: Given $A \subset \mathbb{N}$, if every finite subset of A is in $\mathcal{A}$, then $A \in \mathcal{A}$.

We denote by $\mathcal{A}^{\text{MAX}}$ the maximal elements of $\mathcal{A}$, that is $A \in \mathcal{A}^{\text{MAX}}$ if $A \in \mathcal{A}$ and $B \in \mathcal{A}$ with $A \subseteq B$ implies that $A = B$.

Let c_{00} be the vector space of all finitely supported sequences with standard basis $(e_i)_{i\in\mathbb{N}}$. If $\mathcal{A}$ is adequate and $1 \leq p < \infty$, then $h_{\mathcal{A},p}$ is the completion of c_{00} with respect to the norm

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \sup_{A\in\mathcal{A}} \left(\sum_{i\in A} |a_i|^p \right)^{1/p}.$$

It is clear that $(e_i)_{i\in\mathbb{N}}$ is a normalized 1-unconditional basis for $h_{\mathcal{A},p}$.

If $\mathcal{A}$ is an adequate family of finite sets of $\mathbb{N}$ which is *spreading*, that is, if $\{k_1, \ldots, k_n\} \in \mathcal{A}$ and $k_i \leq l_i$, then $\{l_1, \ldots, l_n\} \in \mathcal{A}$, then $\mathcal{A}$ is often called a *regular* family of subsets of $\mathbb{N}$ and the space $h_{\mathcal{A},1}$ is called a combinatorial Banach space and $h_{\mathcal{A},p}$ is its p-convexification (see e.g. [ABC19]).

From [ALMT21] we need the notion of *minimal norming subsets* for vectors in a Banach space with 1-unconditional basis (see also the notion of *1-sets* in [BDHQ19]).

Definition 5.2.2. For any Banach space X with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$ and for $x \in X$, define

$$M(x) := \{A \subseteq \mathbb{N} : \|P_A x\| = \|x\|, \|P_A x - x_i e_i\| < \|x\|, \text{ for all } i \in A\},$$

and

$$M^\infty(x) := \{A \in M(x) : |A| = \infty\}.$$

For any $B = (b_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$, assume that the elements are ordered, that is, $b_i < b_{i+1}$ for all i. For $n \in \mathbb{N}$ we define $B(n) := (b_i)_{i=1}^n$. Let X be a Banach space with a 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$, if $n \in \mathbb{N}$, then by [ALMT21, Lemma 2.8] the set $\bigcup_{D\in M^\infty(x)} \{D(n)\}$ is finite whenever $x \in S_X$. This means that there exists $s = s(n) \in \mathbb{N}$ and sets $D_1, \ldots, D_s$ in $M^\infty(x)$ such that

$$\bigcup_{D\in M^\infty(x)} \{D(n)\} = \{D_1(n), D_2(n), \ldots, D_s(n)\}.$$

Our next goal is to use [ALMT21, Lemma 2.14] to show that if the number of elements s in this set does not grow too fast as n increases, then x is not a delta-point.

Theorem 5.2.3. *Let X be a Banach space with 1-unconditional basis. If for $x \in S_X$ there exists $n \in \mathbb{N}$ such that for $s = |\bigcup_{D \in M^\infty(x)} \{D(n)\}|$*

$$\|P_E x\| > 1 - \frac{1}{2s} \quad \text{for all} \quad E \in \bigcup_{D \in M^\infty(x)} \{D(n)\},$$

then x is not a delta-point.

In particular, if $|M^\infty(x)| < \infty$, then x is not a delta-point.

Proof. Let X have 1-unconditional basis $(e_i)_{i \in \mathbb{N}}$ and let $x \in S_X$. Since changing signs of the coordinates of vectors is an isometry on spaces with 1-unconditional basis and delta-points are preserved by isometries we may assume that $x = (x_i)_{i \in \mathbb{N}} \in S_X$ with $x_i \geq 0$ for all $i \in \mathbb{N}$. By assumption there exists $n \in \mathbb{N}$, $s \in \mathbb{N}$ and $D_1, \ldots, D_s \in M^\infty(x)$ such that $\|P_{D_k(n)} x\| > 1 - \frac{1}{2s}$ for all $k = 1, \ldots, s$.

For each $k \leq s$ let $x_k^* \in S_{X^*}$ be such that $x_k^*(P_{D_k(n)} x) = \left\|P_{D_k(n)} x\right\|$ and $x_k^*(e_j) = 0$ for all $j \notin D_k(n)$.

Let $x^* = \frac{1}{s} \sum_{k=1}^s x_k^*$. Then $x^*(x) > 1 - \frac{1}{2s}$, so $x \in S(x^*, \frac{1}{2s})$. If $y \in S(x^*, \frac{1}{2s})$ it follows that $x_k^*(y) > 1 - \frac{1}{2}$ for all $k = 1, \ldots, s$. Fix k. If $y_i \leq x_i/2$ for all $i \in D_k(n)$, then $x_k^*(y) \leq \frac{1}{2} x_k^*(x) \leq \frac{1}{2}$. Thus for each $1 \leq k \leq s$, there exists $i \in D_k(n)$ such that $y_i > \frac{1}{2} x_i$ and therefore we can apply [ALMT21, Lemma 2.14] with $S(x^*, \frac{1}{2s})$, n and $\eta = \frac{1}{2}$ and conclude that x is not a delta-point. $\square$

Example 5.2.4. Recall that a norm in a Banach lattice X is called *strictly monotone* if $\|x + y\| > \|x\|$ whenever $x, y \in X$ with $x, y \geq 0$ and $y \neq 0$. From Theorem 5.2.3 we immediately get that a Banach space X with a 1-unconditional basis and such a norm has no delta-points since in this case $M(x) = \{\text{supp}(x)\}$ for any $x \in S_X$.

Example 5.2.5. The space $X = \ell_1 \oplus_\infty \ell_1$ has a 1-unconditional basis. Moreover, it is easily seen that $|M^\infty(x)| \leq 2$ for any $x \in S_X$, so X has no delta-points.

5.3 The case $1 < p < \infty$

In this section we prove the following theorem.

Theorem 5.3.1. *Let $\mathcal{A}$ be an adequate family of subsets of $\mathbb{N}$ and let $1 < p < \infty$. Then*

(i) $h_{\mathcal{A},p}$ does not have delta-points.

(ii) $h_{\mathcal{A},p}^*$ does not have delta-points.

Proof of Theorem 5.3.1 (i). Let $x \in S_{h_{\mathcal{A},p}}$. As in the proof of Theorem 5.2.3 we assume without loss of generality that $x_i \geq 0$ for all i.

As noted above Theorem 5.2.3 we can find $k \in \mathbb{N}$ and $D_j \in M^\infty(x)$ for $j = 1, \ldots, k$ such that

$$\bigcup_{D \in M^\infty(x)} \{D(1)\} = \{D_1(1), \ldots, D_k(1)\}.$$

For each $j = 1, \ldots, k$ define

$$x_j^* = \sum_{i \in D_j} x_i^{p-1} e_i.$$

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölders inequality we have

$$\begin{aligned} |x_j^*(y)| &\leq \sum_{i \in D_j} |x_i^{p-1}||y_i| \leq \left(\sum_{i \in D_j} (|x_i^{p-1}|)^q\right)^{1/q} \left(\sum_{i \in D_j} |y_i|^p\right)^{1/p} \\ &\leq \|x\|^{p/q} \|y\| \leq 1 \end{aligned}$$

As $x_j^*(x) = 1$ we conclude that $x_j^* \in S_{X^*}$.

For each j write $D_j = (d_i^j)_{i \in \mathbb{N}}$ with $d_i^j < d_{i+1}^j$. Define

$$\xi_j := (x_{d_1^j}^{p-1}, x_{d_2^j}^{p-1}, \ldots) \in S_{\ell_q}$$

and $T_j : h_{\mathcal{A},p} \to \ell_p$ by

$$T_j(\sum_{i=1}^\infty a_i e_i) = (a_{d_1^j}, a_{d_2^j}, \ldots).$$

Note that $\|T_j(x)\|_p = 1$ since $D_j \in M^\infty(x)$. Let $\varepsilon := \frac{1}{2} \min_j x_{d_1^j} > 0$. Clarkson [Cla36] showed that ℓ_p is uniformly convex so there exists $\delta_j > 0$ such that if

$$y \in S(\xi_j, \delta_j) \subseteq B_{\ell_p}$$

then $\|T_j(x) - y\|_p < \varepsilon$.

Define $\delta := \min_j \delta_j$ and $x^* := \frac{1}{k} \sum_{j=1}^k x_j^*$. We have $x^*(x) = 1$ and if $z \in S(x^*, \frac{\delta}{k})$, then

$$T_j(z) \in S(\xi_j, \delta) \subseteq S(\xi_j, \delta_j),$$

hence $|x_{d_1^j} - z_{d_1^j}| \leq \|T_j(x) - T_j(z)\| < \varepsilon$. By definition of ε we get $z_{d_1^j} \geq \frac{1}{2} x_{d_1^j} > 0$. Applying [ALMT21, Lemma 2.14] with x^*, δ/k, $\eta = \frac{1}{2}$ and $n = 1$ the result follows. □

For the proof that $h^*_{\mathcal{A},p}$ does not have delta-points we first need to show that the standard basis is shrinking.

Lemma 5.3.2. *Let $\mathcal{A}$ be an adequate family and let $1 < p < \infty$. Then the standard basis $(e_i)_{i\in\mathbb{N}}$ of $h_{\mathcal{A},p}$ is shrinking. In particular, $h^*_{\mathcal{A},p}$ is separable and has an unconditional basis.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized block basis of $(e_i)_{i\in\mathbb{N}}$. This means that there is a sequence $1 \le p_1 < p_2 < \cdots$ and coefficients (a_i^n) such that $x_n = \sum_{i=p_n}^{p_{n+1}-1} a_i^n e_i$ satisfies $\|x_n\| = 1$.

Define an operator $S : \ell_p \to h_{\mathcal{A},p}$ by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n x_n.$$

We have that S is a bounded linear operator. Indeed, for $(\lambda_n) \in \ell_p$ and $A \in \mathcal{A}$ we define $A_n = A \cap \operatorname{supp}(x_n)$. Then we have

$$\sum_{i\in A} |S((\lambda_n))_i|^p = \sum_{n=1}^{\infty} \sum_{i\in A_n} |\lambda_n a_i^n|^p = \sum_{n=1}^{\infty} |\lambda_n|^p \sum_{i\in A_n} |a_i^n|^p \le \sum_{n=1}^{\infty} |\lambda_n|^p \|x_n\|^p$$

and hence $\|S\| \le 1$. If (f_n) denotes the standard basis in ℓ_p, then $S(f_n) = x_n$. Since S is weak–weak continuous we get that (x_n) is weakly null since (f_n) is weakly null in ℓ_p. By [AK06, Proposition 3.2.7] $(e_i)_{i\in\mathbb{N}}$ is shrinking. □

Proof of Theorem 5.3.1 (ii). By Lemma 5.3.2 the standard basis $(e_i)_{i\in\mathbb{N}}$ for $X := h_{\mathcal{A},p}$ is shrinking and hence the biorthogonal functionals $(e_i^*)_{i\in\mathbb{N}}$ is an 1-unconditional basis for X^*. Furthermore, we know that X^{**} is a sequence space and that for $x^{**} = (a_i)_{i\in\mathbb{N}} \in X^{**}$ we have $\|x^{**}\| = \sup_N \|P_N x^{**}\|$ where $P_N(x^{**}) = \sum_{i=1}^{N} a_i e_i \in X$.

Let $x^* = (x_j^*)_{j\in\mathbb{N}} \in S_{X^*}$. Without loss of generality we may assume that $x_j^* \ge 0$ for each j. By Theorem 5.2.3 it is enough to show that $M^\infty(x^*)$ is finite. To this end we will show that $M(x^*) = \{\operatorname{supp}(x^*)\}$.

Assume for contradiction that there exists $j \in \operatorname{supp}(x^*)$ such that $\|x^* - x_j^* e_j^*\| = 1$. Find $x^{**} \in S_{X^{**}}$ such that $x^{**}(x^* - x_j^* e_j^*) = 1$ and $x^{**}(e_k^*) = 0$ for all $k \notin \operatorname{supp}(x^* - x_j^* e_j^*)$. Let $y^{**} = x_j^{*q-1} e_j + (1 - x_k^{*q})^{1/p} x^{**}$. For $A \in \mathcal{A}$ and $N \in \mathbb{N}$ denote

$A_N = A \cap \{1, \ldots, N\}$. We have

$$\begin{aligned}\|P_N y^{**}\|^p &= \sup_{A\in\mathcal{A}} \sum_{i\in A_N} |y_i^{**}|^p \\ &\le (x_j^{*q-1})^p + \left(1 - x_j^{*q}\right) \sup_{A\in\mathcal{A}} \sum_{i\in A_N} |x_i^{**}|^p \\ &\le x_j^{*q} + \left(1 - x_j^{*q}\right) = 1.\end{aligned}$$

This yields $\|y^{**}\| = \sup_N \|P_N y^{**}\| \le 1$ and since $x^* \in S_{X^*}$ we arrive at the contradiction

$$y^{**}(x^*) = x_j^{*q} + \left(1 - x_j^{*q}\right)^{1/p} > x_j^{*q} + \left(1 - x_j^{*q}\right) = 1.$$

Hence we can leave no index behind and $M(x^*) = \{\text{supp}(x^*)\}$. □

5.4 The case $p = 1$

We now turn to $h_{\mathcal{A},1}$ spaces. In this case the situation is not as clear as for $p > 1$. Let us first note that the extreme points of the dual space has a well-known characterization.

Lemma 5.4.1 (Lemma 2.3 in [AM93])**.** *Let $\mathcal{A}$ be an adequate family of $\mathbb{N}$. Then*

$$\text{ext}\, B_{h^*_{\mathcal{A},1}} = \left\{ \sum_{i\in A} \varepsilon_i e_i^* : A \in \mathcal{A}^{MAX}, \varepsilon_i \in \{-1, 1\} \right\}.$$

In [ALMT21, Theorem 3.1] it was shown that if we build an adequate family on $\mathbb{N}$ by using the branches of the binary tree, then we get an $h_{\mathcal{A},1}$ space with delta-points. This adequate family contains many infinite sets. We have the following general result about duals of $h_{\mathcal{A},1}$ spaces when the adequate family contains an infinite set.

Proposition 5.4.2. *Let $\mathcal{A}$ be an adequate family that contains an infinite set. Then there exists $x^* \in h^*_{\mathcal{A},1}$ such that x^* is an extreme point and a delta-point.*

Proof. As $\mathcal{A}$ contains an infinite set there exist $A \in \mathcal{A}^{\text{MAX}}$ with $|A| = \infty$. Write $A = (a_i)_{i=1}^\infty$ with $a_i < a_{i+1}$ for all i. For $x^{**} \in S_{h^{**}_{\mathcal{A},1}}$ we have $y^* = \sum_{i=1}^\infty \text{sgn}(x^{**}(e^*_{a_i}))e^*_{a_i} \in B_{h^*_{\mathcal{A},1}}$ by Lemma 5.4.1. Hence

$$\sum_{i=1}^\infty |x^{**}(e^*_{a_i})| = x^{**}(y^*) \le 1$$

and this implies that $(e^*_{a_i})_{i=1}^\infty$ is weakly null.

Now let $x^* = \sum_{i\in A} e^*_i \in \operatorname{ext} B_{h^*_{\mathcal{A},1}}$. Let $x^{**} \in S_{h^{**}_{\mathcal{A},1}}$ and $\varepsilon > 0$ with $x^* \in S(x^{**}, \varepsilon)$. Define $x^*_i = x^* - 2e^*_{a_i} \in \operatorname{ext} B_{h^*_{\mathcal{A},1}}$ so that $\|x^* - x^*_i\| = \|2e^*_{a_i}\| = 2$. For i large enough we have $x^*_i \in S(x^{**}, \varepsilon)$ and thus x is a delta-point. □

We shall henceforth exclusively consider $h_{\mathcal{A},1}$ spaces where the adequate family consists of finite sets only. Recall that a Banach spaces is said to be *polyhedral* if the unit ball of each of its finite-dimensional subspaces is a polyhedron [Kle60].

The following result is well-known but we include a proof for easy reference.

Proposition 5.4.3. *Let $\mathcal{A}$ be an adequate family of subsets of $\mathbb{N}$. The following are equivalent:*

(i) $\mathcal{A}$ consists of finite sets only;

(ii) $h_{\mathcal{A},1}$ is polyhedral;

(iii) The standard basis $(e_i)_{i\in\mathbb{N}}$ is shrinking.

Proof. (i) $\implies$ (ii) Note that the proof of Theorem 4.5 (see also Remark 4.4) [ABC19] also holds for adequate families of finite sets, not just for regular families, so that $h_{\mathcal{A},1}$ is (V)-polyhedral in the sense Fonf and Veselý [FV04]. In particular, $h_{\mathcal{A},1}$ is polyhedral.

(ii) $\implies$ (iii) If $h_{\mathcal{A},1}$ is polyhedral, then it is c_0 saturated by [Fon80] (see the remark following Theorem 5). Hence $h_{\mathcal{A},1}$ cannot contain an isomorphic copy of ℓ_1 and this yields that $(e_i)_{i\in\mathbb{N}}$ is shrinking since it is unconditional.

(iii) $\implies$ (i). Assume $\mathcal{A}$ contains an infinite set A. Then the basis vectors $(e_i)_{i\in A}$ span an isometric copy of ℓ_1 in $h_{\mathcal{A},1}$ and $(e_i)_{i\in\mathbb{N}}$ is not shrinking. □

We will have more to say about polyhedrality in $h_{\mathcal{A},1}$ spaces below in Section 5.5.

Next we note that extreme points in $h^*_{\mathcal{A},1}$ are actually w^*-exposed.

Proposition 5.4.4. *Let $\mathcal{A}$ be an adequate family of subsets of $\mathbb{N}$ and $x^* \in \operatorname{ext} B_{h^*_{\mathcal{A},1}}$. Then the following are equivalent:*

(i) x^* is an extreme point of $B_{h^*_{\mathcal{A},1}}$;

(ii) x^* is a w^*-exposed point of $B_{h^*_{\mathcal{A},1}}$.

In particular, if $\mathcal{A}$ is an adequate family of finite sets, then x^* is an extreme point if and only if x^* is a w^*-strongly exposed point.

Remark 5.4.5. We showed above in Proposition 5.4.2 that if the adequate family $\mathcal{A}$ contains an infinite set, then there is an extreme point in $h^*_{\mathcal{A},1}$ which is also a delta-point. Proposition 5.4.4 shows that this extreme point is also exposed, but being a delta-point it is far from being a w^*-strongly exposed point.

Proof. One direction is trivial, so we only need to show (i) $\implies$ (ii). Let $x^* \in \operatorname{ext} B_{h^*_{\mathcal{A},1}}$. By Lemma 5.4.1 we can write $x^* = \sum_{i\in B} \varepsilon_i e_i^*$ with $B \in \mathcal{A}^{\text{MAX}}$ and $\varepsilon_i \in \{-1, 1\}$. Find some $y \in S_{\ell_1}$ with supp$(y) = B$ and $y_i \geq 0$ for all $i \in \mathbb{N}$. Define $x = \sum_{i\in B} y_i \varepsilon_i e_i \in S_{h_{\mathcal{A},1}}$ and notice that $x^*(x) = 1$.

Take some $y^* \in S_{h^*_{\mathcal{A},1}}$ with $y^*(x) = 1$. Then $y_i^* = y^*(e_i) = \varepsilon_i$ for all $i \in B$. For $j \in \mathbb{N} \setminus B$, there must exist, by compactness of $\mathcal{A}$, some finite $A \subset B$ such that $A \bigcup \{j\} \notin \mathcal{A}$. Define

$$\tilde{x} = x + \operatorname{sgn} y_i^* \min_{i\in A} |x_i| e_j,$$

and observe that $\|\tilde{x}\| = 1$. Indeed, take $C \in \mathcal{A}$ with $j \in C$. Then $A \setminus C \neq \emptyset$, and so

$$\sum_{i\in C} |\tilde{x}_i| = \sum_{i\in C\cap B} |\tilde{x}_i| + |\tilde{x}_j| \leq \sum_{i\in B} |x_i| - \min_{i\in A} |x_i| + \min_{i\in A} |x_i| = 1.$$

We now get

$$1 \geq y^*(\tilde{x}) = y^*(x) + \min_{i\in A} |x_i||y_j^*| = 1 + \min_{i\in A} |x_i||y_j^*|,$$

implying that $y_j^* = 0$ for all $j \in \mathbb{N} \setminus B$. That is, $y^* = x^*$.

If $\mathcal{A}$ consists of finite sets, then $h_{\mathcal{A},1}$ is polyhedral and for polyhedral spaces the w^*-exposed points and w^*-strongly exposed points of the dual unit ball coincides ([Fon00, Theorem 1.4]). □

Let $\mathcal{A}$ be an adequate family and let $x \in S_{h_{\mathcal{A},1}}$. Since $\mathcal{A}$ is compact in the topology of pointwise convergence in $\mathbb{N}$ there exists $A \in \mathcal{A}$ such that $\|P_A x\| = \|x\|$. In particular, $M(x) \subset \mathcal{A}$. The following result is now immediate from Theorem 5.2.3 (or [ALMT21, Proposition 2.15]).

Proposition 5.4.6. *If $\mathcal{A}$ is an adequate family of finite subsets of $\mathbb{N}$, then $h_{\mathcal{A},1}$ does not have delta-points.*

Definition 5.4.7. A Banach space X has the *convex series representation property* (CSRP) if for each $x \in B_X$, there exists a sequence (f_i) of extreme points of B_X and a sequence of nonnegative real numbers (λ_i) such that $\sum_{i=1}^{\infty} \lambda_i = 1$ and

$$x = \sum_{i=1}^{\infty} \lambda_i f_i.$$

Note that the CSRP is equivalent to the λ-property [ALS91]. The proof of [ABC19, Proposition 4.3] also holds for adequate families of finite sets and thus we have the following.

Proposition 5.4.8 (Proposition 4.3 in [ABC19])**.** *Let $\mathcal{A}$ be an adequate family of finite subsets of $\mathbb{N}$. Then $h^*_{\mathcal{A},1}$ has the CSRP.*

Using this proposition we are able to show that for regular families $\mathcal{A}$ the dual of $h_{\mathcal{A},1}$ does not have delta-points.

Proposition 5.4.9. *If $\mathcal{A}$ is an adequate family of finite subsets of $\mathbb{N}$ which is spreading, then $h^*_{\mathcal{A},1}$ fails to have any delta-points.*

Proof. By assumption the standard basis $(e_i)_{i\in\mathbb{N}}$ for $h_{\mathcal{A},1}$ is shrinking and hence the biorthogonal functionals $(e_i^*)_{i\in\mathbb{N}}$ is a 1-unconditional basis for $h^*_{\mathcal{A},1}$.

We only need to consider x^* with infinite support by Theorem 5.2.3. From Lemma 5.4.1 we have that elements of $\operatorname{ext} h^*_{\mathcal{A},1}$ can be written

$$x^*_{A,(\varepsilon_i)} = \sum_{i\in A} \varepsilon_i e_i^* \quad \text{where} \quad A \in \mathcal{A}^{\text{MAX}} \text{ and } (\varepsilon_i) \in \{-1, 1\}.$$

By Proposition 5.4.8 we can write $x^* = \sum_{n=1}^{\infty} \lambda_n x^*_{F_n,(\varepsilon_i^n)}$ where for $F_n \in \mathcal{A}$ and $\varepsilon_i^n \in \{-1, 1\}$. There are two possibilities. Either $\|x^* - x_k^* e_k^*\| < 1$, for every $k \in \operatorname{supp}(x^*)$, and then $M^\infty(x^*) = \{\operatorname{supp}(x^*)\}$, and the result follows from Theorem 5.2.3. The other possibility is that there exists $k \in \operatorname{supp}(x^*)$ such that $\|x^* - x_k^* e_k^*\| = 1$. Write

$$y^* := x^* - x_k^* e_k^* = \sum_{n=1}^{\infty} \lambda_n x^*_{F_n',(\varepsilon_i^n)},$$

where $F_n' = F_n \setminus \{k\}$ for all n. Find $y^{**} = (y_i) \in S_{X^{**}}$ such that

$$1 = y^{**}(y^*) = \sum_{n=1}^{\infty} \lambda_n y^{**}(x^*_{F_n',(\varepsilon_i^n)}).$$

This implies that $y^{**}(x^*_{F_n',(\varepsilon_i^n)}) = 1$ for every $n \in \mathbb{N}$. Now find $n \in \mathbb{N}$ with $k \in F_n$. Take any integer $j \in (F_n')^C$ with $j \geq \max F_n$, let $G_n := F_n' \cup \{j\}$, and let $\varepsilon_j = \operatorname{sgn} y_j$. As $\mathcal{A}$ is spreading $G_n \in \mathcal{A}$ and thus $\|x^*_{G_n,(\varepsilon_i^n)}\| = 1$. But then,

$$1 \geq y^{**}(x^*_{G_n,(\varepsilon_i^n)}) = y^{**}(x^*_{F_n',(\varepsilon_i^n)}) + y^{**}(\varepsilon_j e_j^*) = 1 + |y_j| \geq 1,$$

which forces y_j to be zero. From this it follows that $y^{**} = (y_i)$ is zero on all coordinates $i \geq \max F_n$ (at least). That is, every $A \in M(x^*)$ must be a subset of $\{1, \ldots, \max F_n\}$, so $M^\infty(x^*) = \emptyset$ and the result follows from Theorem 5.2.3. □

If $\mathcal{A}$ is an adequate family of finite sets which is not spreading, we do not know whether or not $h^*_{\mathcal{A},1}$ can have delta-points. But we are able to say something about Daugavet-points.

Let X be a Banach space. Recall that $x \in S_X$ is a Daugavet-points if and only if for every slice $S(x^*, \delta)$ of B_X and for every $\varepsilon > 0$ there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$ (see e.g. [AHLP20, Lemma 2.3]).

Proposition 5.4.10. *If $x^* \in S_{X^*}$ can be written as*

$$x^* = \sum_{i=1}^{\infty} \lambda_i f_i$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{\infty} \lambda_i = 1$, f_i are (w^*-)strongly exposed points in B_{X^*}, then x^* is not a Daugavet-point.

In particular, if $\mathcal{A}$ is an adequate family of finite sets, then $h^*_{\mathcal{A},1}$ does not have Daugavet-points.

Proof. Choose j with $\lambda_j > 0$. Find a slice $S(x^{**}, \delta)$ containing f_j with diameter less than 1. Then for any $y^* \in S(x^{**}, \delta)$ we have

$$\begin{aligned}
\|x^* - y^*\| &\leq \|\sum_{i \neq j} \lambda_i f_i\| + \|y^* - \lambda_j f_j\| \\
&\leq \sum_{i \neq j} \lambda_i + (1 - \lambda_j) + \lambda_j \|y^* - f_j\| \\
&< 2(1 - \lambda_j) + \lambda_j = 2 - \lambda_j.
\end{aligned}$$

So the distance from x^* to y^* is bounded away from 2. □

5.5 Polyhedrality

In this section we study polyhedrality in $h_{\mathcal{A},1}$ spaces. The goal is to describe the polyhedrality of $h_{\mathcal{A},1}$ spaces in terms of the structure of the adequate family $\mathcal{A}$. Let us begin by recalling some concepts and results.

If X is a Banach space and $A \in X^*$ then A' denotes the set of all *w^*-limit points* of A, that is

$$A' = \left\{ f \in X^* \ : \ f \in \overline{A \setminus \{f\}}^{w^*} \right\}.$$

In [FV04] Fonf and Veselý identified eight known definitions of polyhedrality from the literature and gave examples showing that in general they are different. We will use the following three definitions from their paper.

Definition 5.5.1. Let X be a Banach space. Then

(i) X is (I)-polyhedral if $(\text{ext}\, B_{X^*})' \subseteq \{0\}$;

(ii) X is (IV)-polyhedral if $f(x) < 1$ whenever $x \in S_X$ and $f \in (\text{ext}\, B_{X^*})'$;

(iii) X is (V)-polyhedral if $\sup\{f(x) \,:\, f \in \text{ext}\, B_{X^*} \setminus D(x)\} < 1$ for each $x \in S_X$, where $D(x) = \{f \in S_{X^*} : f(x) = 1\}$.

In the proof of Proposition 5.4.3 we met the following:

Theorem 5.5.2 (Theorem 4.5 in [ABC19])**.** *Let $\mathcal{A}$ be an adequate family of finite subsets of $\mathbb{N}$. Then $h_{\mathcal{A},1}$ is (V)-polyhedral.*

It is natural to ask whether any (V)-polyhedral $h_{\mathcal{A},1}$ can be (IV)- or even (I)-polyhedral. Our next goal is to show the following: If $h_{\mathcal{A},1}$ is polyhedral, then it is either (I)-polyhedral or (V)-polyhedral. Considering Proposition 5.4.3 and Theorem 5.5.2 we only need to show the following.

Theorem 5.5.3. *Let $\mathcal{A}$ be an adequate family of finite sets. Then the following are equivalent:*

(i) $\{A \in \mathcal{A} \,:\, i \in A\}$ *is finite for all* $i \in \mathbb{N}$.

(ii) $h_{\mathcal{A},1}$ *is (I)-polyhedral.*

(iii) $h_{\mathcal{A},1}$ *is (IV)-polyhedral.*

Proof. (ii) $\implies$ (iii) is trivial.

(i) $\implies$ (ii). By assumption and Lemma 5.4.1, for any $i \in \mathbb{N}$, there are only a finite number of extreme points that have support on i. It follows that if $f \in (\text{ext}\, B_{X^*})'$, then $f(e_i) = 0$ for all $i \in \mathbb{N}$. Therefore f is 0.

(iii) $\implies$ (i). First observe that if $\{A \in \mathcal{A} \,:\, i \in A\}$ is infinite, then the set $\mathcal{C}_i = \left\{A \in \mathcal{A}^{\text{MAX}} \,:\, i \in A\right\}$ is also infinite.

Assume that for $i \in \mathbb{N}$ the set $\mathcal{C}_i$ is infinite and let $(A_n)_{n=1}^\infty \subset \mathcal{C}_i$ be a sequence of distinct sets. As elements of $\mathcal{A}$ are finite and $\mathcal{A}$ is compact in the topology of pointwise convergence on $\mathbb{N}$, we can by passing to a subsequence if necessary assume that (A_n) converges pointwise to some $A \in \mathcal{A}$ and that $A \subset A_n$ for all $n \in \mathbb{N}$.

If $k \in \mathbb{N}$ then there exists $N \in \mathbb{N}$ such that $\{1, 2, \dots, k\} \cap A_n = \{1, 2, \dots, k\} \cap A$ for all $n > N$, i.e. $\min(A_n \setminus A) \to \infty$.

With the sequence (A_n), we wish to show that $h_{\mathcal{A},1}$ is not (IV)-polyhedral. That is, we wish to show that there exist some $x^* \in (\operatorname{ext} B_{h^*_{\mathcal{A},1}})'$ and some $x \in S_{h_{\mathcal{A},1}}$ such that $x^*(x) = 1$. To this end, define $x^* = \sum_{i \in A} e_i^*$ and $x = \frac{1}{|A|} \sum_{i \in A} e_i \in S_{h_{\mathcal{A},1}}$. As $x^*(x) = 1$, it remains to show that x^* is a w^*-limit of elements in $\operatorname{ext} B_{h^*_{\mathcal{A},1}} \setminus \{x^*\}$.

By Lemma 5.4.1 $x_n^* := \sum_{i \in A_n} e_i^* = x^* + \sum_{i \in A_n \setminus A} e_i^* \in \operatorname{ext} B_{h^*_{\mathcal{A},1}}$. If $y \in h_{\mathcal{A},1}$ we get that

$$\begin{aligned} |x_n^*(y) - x^*(y)| &= |x^*(y) + \sum_{i \in A_n \setminus A} y_i - x^*(y)| \\ &\leq \sum_{i \in A_n \setminus A} |y_i|. \end{aligned}$$

Since $\sum_{i \in A_n \setminus A} |y_i| \to 0$ because $\min(A_n \setminus A) \to \infty$ we get $x_n^* \xrightarrow{w^*} x^*$ and thus $h_{\mathcal{A},1}$ is not (IV)-polyhedral. □

The above theorem shows that the only combinatorical Banach space which is (IV)-polyhedral is c_0. Antunes, Beanland and Chu [ABC19, Theorem 4.5] (see also their Remark 4.4) proved that combinatorial Banach spaces are (V)-polyhedral. So our next corollary shows that this is in fact best possible.

Corollary 5.5.4. *If $\mathcal{A}$ is a regular family of sets such that $h_{\mathcal{A},1}$ is (IV)-polyhedral, then $h_{\mathcal{A},1} = c_0$.*

Proof. Assume that $A \in \mathcal{A}$ with $|A| > 1$. Let $i = \min A$. Since there are infinitely many spreads of A we have that $\{A \in \mathcal{A} : i \in A\}$ is infinite. □

We end the paper with some questions.

Let $\mathcal{A}$ be an adequate family. Proposition 5.4.3 gives a simple characterization of when $h_{\mathcal{A},1}$ contains a copy of ℓ_1 and $h_{\mathcal{A},p}$ never contains ℓ_1 by Lemma 5.3.2 (and James' characterization of shrinking unconditional bases).

Note that $h_{\mathcal{A},1}$ and even $h_{\mathcal{A},p}$ contains an isometric copy of c_0 if there exists an infinite subset $E \subseteq \mathbb{N}$ such that $|A \cap E| \leq 1$ for all $A \in \mathcal{A}$. We ask:

Question 3. What is a natural condition on $\mathcal{A}$ such that $h_{\mathcal{A},1}$ (or $h_{\mathcal{A},p}$) does not contain c_0. That is, when does $h_{\mathcal{A},p}$ have a boundedly complete basis?

We have seen that if $\mathcal{A}$ is an adequate family of finite sets which is spreading, then neither $h_{\mathcal{A},1}$ nor $h^*_{\mathcal{A},1}$ have delta-points. We do not know if that is also the case if $\mathcal{A}$ is not spreading.

Question 4. If $h_{\mathcal{A},1}$ is (V)-polyhedral, does then $h^*_{\mathcal{A},1}$ fail to have delta-points?

One can even ask: If $h_{\mathcal{A},1}$ is (I)-polyhedral, does then $h^*_{\mathcal{A},1}$ fail to have delta-points? Note that if X is (I)-polyhedral, then X is isometric to a subspace of c_0 [FV04, Theorem 1.2].

Acknowledgement

The authors would like to thank Stanimir Troyanski for discussions on the topic of the paper.

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6 Appendix

In this Appendix we tie up some loose ends concerning Müntz spaces and prove special cases that were not covered in the papers forming Chapters 2–3. For notation and terminology, see Chapter 1.

In Proposition 3.2.3 it was shown that $M(\Lambda)$ is almost isometric to a subspace of c whenever $\lambda_1 \geq 1$. It is natural to think that Proposition 3.2.3 could be extended to hold for any Müntz space. Our first goal is to show that this is indeed the case. For the following results, unless explicitly stated otherwise, Λ will be any Müntz sequence with $\lambda_1 < 1$ and $k \in \mathbb{N}$ such that $\lambda_1 < \lambda_2 < \ldots < \lambda_k < 1 \leq \lambda_{k+1}$.

The next lemma follows by studying the proof of [GL05, Corollary 6.1.3].

Lemma 6.0.1. *For any $M(\Lambda)$ Müntz space and $m \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that if $g = \sum_{i=0}^{m} a_i t^{\lambda_i} + \sum_{i=m+1}^{n} a_i t^{\lambda_i} \in \Pi(\Lambda)$, then $|a_l| < M \|g\|$ for all $l \in \{0, 1, \ldots, m\}$.*

Using this Lemma, we also get the following.

Lemma 6.0.2. *Let $M(\Lambda)$ be a Müntz space. For any $0 < a < b < 1$, there exists a constant $C_{a,b}$ such that*

$$\left\|g'\right\|_{[a,b]} < C_{a,b} \|g\|,$$

for any $g \in \Pi(\Lambda)$.

Proof. Start with $0 < a < b < 1$ and let M be the constant from Lemma 6.0.1. By the Bounded Bernstein inequality (see Theorem 3.2.1) there is a constant c such that $\|p'\|_{[0,b]} < c\|p\|$ for any $p \in \text{span}\left\{t^{\lambda_i} \,:\, i > k\right\}$. For $g \in \Pi(\Lambda)$ and $a \leq t \leq b$ we have that $a^{\lambda_1 - 1} \geq t^{\lambda_i - 1}$ for all $i \in \{1, \ldots, k\}$, thus

$$\begin{aligned}
\left\|g'\right\|_{[a,b]} &\leq \sum_{i=1}^{k} \left\|a_i \lambda_i t^{\lambda_i - 1}\right\|_{[a,b]} + \|g' - \sum_{i=1}^{k} a_i \lambda_i t^{\lambda_i - 1}\|_{[a,b]} \\
&< k \cdot \lambda_k \cdot a^{\lambda_1 - 1} \cdot M \|g\| + c\|g - \sum_{i=1}^{k} a_i t^{\lambda_i}\| \\
&\leq k \cdot M \cdot a^{\lambda_1 - 1} \|g\| + c\left(\|g\| + k \cdot M \|g\|\right) \\
&= \left(k \cdot M a^{\lambda_1 - 1} + c(1 + kM)\right) \|g\| < \infty.
\end{aligned}$$

We now see that $\|g'\| \leq C_{a,b}\|g\|$, where $C_{a,b} = \big(k \cdot Ma^{\lambda_1 - 1} + c(1+kM)\big)$. □

Lemma 6.0.3. *Let $M(\Lambda)$ be a Müntz space. Then for any $\varepsilon > 0$ there exists $s > 0$ such that $|f(t) - f(0)| < \varepsilon \|f\|$ for all $t < s$ and $f \in M(\Lambda)$.*

Proof. Let $\varepsilon > 0$ and let M be the constant from Lemma 6.0.1. By the Bounded Bernstein inequality (see Theorem 3.2.1) there is a constant c such that $\|p'\|_{[0,1/2]} < c\|p\|$ for any $p \in \text{span}\left\{t^{\lambda_i} : i \in \{k+1, k+2, \ldots\} \cup \{0\}\right\}$. Pick $s \in (0, 1/2)$ such that $s^{\lambda_1} < \frac{\varepsilon}{(kM+c(1+kM))}$. For $g = \sum_{i=0}^{n} a_i t^{\lambda_i}$ set $h = \sum_{i=1}^{k} a_i t^{\lambda_i}$. Then for any $\alpha \in [0,1]$

$$\|h\|_{[0,\alpha]} \leq \sum_{i=1}^{k} |a_i|\alpha^{\lambda_i} \leq kM\|g\|\alpha^{\lambda_1}.$$

We get that for any $t \leq s$

$$\begin{aligned}
|g(t) - g(0)| &= |h(t) + (g-h)(t) - (g-h)(0)| \\
&\leq \|h\|_{[0,s]} + |t-0|\|(g-h)'\|_{[0,\frac{1}{2}]} \\
&\leq kM\|g\|s^{\lambda_1} + sc\|g-h\|_{[0,1]} \\
&\leq kM\|g\|s^{\lambda_1} + sc(\|g\| + kM\|g\|) \\
&\leq s^{\lambda_1}(kM + c(1+kM))\|g\|
\end{aligned}$$

As $\Pi(\Lambda)$ is dense in $M(\Lambda)$ the result follows. □

We are now ready to extend Proposition 3.2.3 to include all Müntz spaces. The proof is just a modification of the original argument.

Proposition 6.0.4. *Let $M(\Lambda)$ be a any Müntz space. Then for any $\varepsilon > 0$ there exists an operator $J_\varepsilon : M(\Lambda) \to c$ such that*

$$(1-\varepsilon)\|f\| \leq \|J_\varepsilon f\| \leq \|f\|.$$

Proof. By Proposition 3.2.3 we only need to consider $M(\Lambda)$ where $\lambda_1 < \lambda_2 < \ldots < \lambda_k < 1 \leq \lambda_{k+1}$. Let $\varepsilon > 0$ and from Lemma 6.0.3 find $s_0 \in (0,1)$ such that $|f(t) - f(0)| < \varepsilon\|f\|$ for all $t < s_0$ and $f \in M(\Lambda)$. Now pick a sequence $s_0 = a_0 < a_1 < a_2 < \ldots < a_i < \ldots$ converging to 1. By Lemma 6.0.2, we can for each a_i, find C_{s_0,a_i} such that for any $g \in \Pi(\Lambda)$ we get

$$\left\|g'\right\|_{[s_0,a_i]} \leq C_{s_0,a_i}\|g\|,$$

with $i \in \mathbb{N}$. Next we pick points $s_0 = a_0 < s_1 < \ldots < s_{n_1} = a_1 < s_{n_1+1} < \ldots < s_{n_2} = a_2 < \ldots$ such that

$$s_{j+1} - s_j \leq \frac{\varepsilon}{C_{s_0,a_{i+1}}} \text{ for } n_i \leq j \leq n_{i+1}.$$

Define the operator $J_\varepsilon : M(\Lambda) \to c$ by $J_\varepsilon(f) = (f(0), f(s_0), f(s_1), f(s_2), \ldots)$. Then by continuity of f the operator is well defined, and $\|J_\varepsilon f\| = \sup_{n\in\mathbb{N}} |f(s_n)| \leq \|f\|$ for all $f \in M(\Lambda)$. Now, let $f \in M(\Lambda)$ and let $g \in \Pi(\Lambda)$ be such that $\|f - g\| < \delta$. If $0 \leq s \leq s_0$, we get from Lemma 6.0.3 that

$$\begin{aligned} |f(s)| \leq |g(s)| + \delta &\leq |g(s) - g(0)| + |g(0)| + \delta \\ &\leq \varepsilon \|g\| + (|f(0)| + \delta) + \delta \\ &\leq \varepsilon (\|f\| + \delta) + \|J_\varepsilon f\| + 2\delta. \end{aligned}$$

Rearranging terms yields

$$|f(s)| - \varepsilon \|f\| \leq \|J_\varepsilon f\| + \delta(\varepsilon + 2).$$

If $s_0 < s < 1$, we have that $a_i \leq s < a_{i+1}$ for some $i \in \mathbb{N}$. Let $s_m \in [a_i, a_{i+1}]$ be such that $|s - s_m| \leq \frac{\varepsilon}{C_{s_0,a_{i+1}}}$. Then

$$\begin{aligned} |f(s)| \leq |g(s)| + \delta &\leq |g(s) - g(s_m)| + |g(s_m)| + \delta \\ &\leq \left\|g'\right\|_{[a_i,a_{i+1}]} |s - s_m| + \|J_\varepsilon g\| + \delta \\ &\leq \|g\| C_{s_0,a_{i+1}} \frac{\varepsilon}{C_{s_0,a_{i+1}}} + \|J_\varepsilon g\| + \delta \\ &\leq \|g\|\varepsilon + \|J_\varepsilon g\| + \delta \\ &\leq (\|f\| + \delta)\varepsilon + (\|J_\varepsilon f\| + \delta) + \delta \end{aligned}$$

and therefore

$$(1 - \varepsilon)\|f\| \leq \|J_\varepsilon f\| + \delta(\varepsilon + 2).$$

Since δ was arbitrary we conclude that

$$(1 - \varepsilon)\|f\| \leq \|J_\varepsilon f\| \leq \|f\|. \qquad \square$$

In *"On Octahedrality and Müntz spaces"* the following result was shown:

Proposition 3.3.2: [Mar]

No Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$ is LASQ.

The proof relied on an upper bound, c, similar to the one in Lemma 6.0.3. In fact, with Lemma 6.0.3 a straightforward modification of the proof of Proposition 3.3.2 yields the following.

Theorem 6.0.5. *No Müntz space $M_0(\Lambda)$ or $M(\Lambda)$ is LASQ.*

Proof. By Lemma 6.0.3 we only need to consider $M(\Lambda)$ where $\lambda_1 < \lambda_2 < \ldots < \lambda_k < 1 \leq \lambda_{k+1}$. By considering the constant 1 function, we can see that $M(\Lambda)$ is never LASQ. Now, consider a Müntz space $M_0(\Lambda)$. By Lemma 6.0.3 we can choose some $s \in (0,1)$ such that $|f(t) - f(0)| = |f(t)| < 1/4$ for all $t < s$ and for all $f \in S_{M(\Lambda)}$.

Recall from [ALL16, Theorem 2.1] that $M_0(\Lambda)$ is LASQ if and only if for every $g \in S_{M(\Lambda)}$ and $\varepsilon > 0$ there exists $h \in S_{M(\Lambda)}$ such that $\|g \pm h\| \leq 1 + \varepsilon$. We claim that no such h exists for $g = t^{\lambda_1}$. Indeed, if $0 < \varepsilon < s^{\lambda_1}/2$ and $h \in S_{\Pi(\Lambda)}$ is such that $\|t^{\lambda_1} \pm h\| \leq 1 + \varepsilon$. Then $|h(t)| < 1 - \varepsilon$ for $t \geq s$ as $t^{\lambda_1} > 2\varepsilon$ for $t \geq s$. Thus, h must attain its norm on the interval $[0, s]$, contradicting our observation. As $\Pi((\lambda_n)_{n=1}^{\infty})$ is dense in $M_0(\Lambda)$, we conclude that $M_0(\Lambda)$ is not LASQ. $\square$

In Chapter 4 Proposition 4.2.12 the following result was shown:

Proposition 4.2.12: [ALMT]

Let X be a Banach space with 1-unconditional basis $(e_i)_{i\in\mathbb{N}}$. If $x \in S_X$, then there exist $\delta > 0$ and a relatively weakly open subset W, with $x \in W$, such that $\sup_{y\in W} \|x - y\| < 2 - \delta$.

Let us show that if f is a Daugavet-point in a Müntz space $M(\Lambda)$, then the above proposition does not hold for f. In fact, we show more:

Proposition 6.0.6. *Let $M(\Lambda)$ be any Müntz space, and $f \in S_{M(\Lambda)}$ with $|f(1)| = 1$. Then whenever C is a convex combination of non-empty relatively weakly open subsets of the unit ball we have*

$$\sup_{g\in C} \|g - f\| = 2.$$

Proof. Let $X = M(\Lambda)$, $f \in S_X$ with $f(1) = 1$, and let C be a convex combination of non-empty relatively weakly open subsets of B_X. Let $(\lambda_k)_{k=1}^{\infty}$ be a RIP subsequence of Λ, and let p_k be the corresponding spike-function to $(\lambda_k)_{k=1}^{\infty}$ and x_k be the unique point where p_k attains its norm (see Lemma 2.2.3 and Remark 2.2.4). Pick any $g = \sum_{i=1}^{n} \mu_i g^i \in C$, and define

$$\tilde{g}_k^i = g^i - \left(1 + g^i(x_k)\right) \frac{p_k}{\|p_k\|}.$$

By the calculations in Theorem 2.2.5, we get that $\left\|\tilde{g}_k^i\right\| \to 1$, that $(\tilde{g_k}^i)$ is bounded and converges pointwise to g^i, and consequently $\tilde{g_k}^i$ converges weakly to g^i. Thus $h_k^i = \frac{\tilde{g}_k^i}{\|\tilde{g}_k^i\|}$ also converges weakly to g^i and thus $\sum_{i=1}^{n} \mu_i h_k^i$ is eventually in C.

It remains to show that $\left\|\sum_{i=1}^{n} \mu_i h_k^i - f\right\| \to 2$. However, as $f(1) = 1$, $h_k^i(x_k) \to -1$, and $x_k \to 1$, this follows from continuity of f.

Clearly a similar argument for $f \in S_X$ with $f(1) = -1$ will work. □

Daugavet- and delta-points of certain Müntz spaces were characterized in Theorem 3.13 [AHLP20]. Again Lemma 6.0.2 and Lemma 6.0.3 can be used to generalize this result to all Müntz spaces.

Theorem 6.0.7. *For any Müntz space $M_0(\Lambda)$ or $M(\Lambda)$, the following assertions for $f \in S_X$ are equivalent:*

(i) *f is a Daugavet-point;*

(ii) *f is a delta-point;*

(iii) $\|f\| = |f(1)|$.

Proof. By [AHLP20, Theorem 3.13], we only consider $M(\Lambda)$ where $\lambda_1 < \lambda_2 < \ldots < \lambda_k < 1 \leq \lambda_{k+1}$.

$(i) \implies (ii)$. This is trivial.

$(ii) \implies (iii)$. Assume for a contradiction that there exists $f \in M(\Lambda)$, with $|f(1)| < 1$. Assume $|f(0)| = 1$ and use Lemma 6.0.3 to find $q \in (0,1)$ such that $\|g - g(0)\|_{[0,q]} < 1/2$ for all $g \in S_{\Pi(\Lambda)}$. As f is the restriction of an analytic function on $\{z \in \mathbb{C} \setminus [-1,0] : |z| < 1\}$ to $(0,1)$, we can conclude that $I = \{t \in (q,1) : |f(t)| = 1\} \cup \{0\}$ is a finite set, by the Principle of permanence. Note that the Principle of permanence also allows us to assume that $|f(q)| < 1$ (otherwise $f(s) = 1$ for all $s < q$ implying that $f = 1$ for all $s \in (0,1)$).

Let $d = \frac{\max I + 1}{2}$ and $t_0 = \frac{\min I \setminus \{0\} + q}{2} > 0$ and $I_0 = [0,q]$. Let $C = C_{t_0,d}$ be the constant from Lemma 6.0.2 and assume that $0 < \gamma < \frac{1}{2C}$. Define $I_t = (t-\gamma, t+\gamma)$. Choose γ smaller if necessary such that $I_t \cap I_s = \emptyset$ whenever $t, s \in I$ and $t \neq s$. By continuity of f and by reducing γ further if necessary, we can assume that

$$1 - |f(s)| < 1/4 \text{ whenever } s \in I_t \text{ and } t \in I \setminus \{0\}.$$

Thus, by defining

$$\eta := 1 - \sup \left\{ |f(s)| \ : \ s \in [0,1] \setminus \big(\bigcup_{t \in I} I_t \big) \right\},$$

we get that $\eta < 1/2$. Suppose $g \in B_X$ and $\|g - f\| > 2 - \eta$. Then there exists $j \in I$ and $s \in I_j$ with $\operatorname{sgn} g(s) = -\operatorname{sgn} f(s)$, and with $|g(s)| > 1 - \eta$. Therefore we cannot have $t \in I_j$ with $|g(j)| > 1 - \eta$ and $\operatorname{sgn} g(j) = -\operatorname{sgn} g(s)$ as this would contradict that $\|g'\|_{[t_0,d]} < T$ if $j > 0$ and our choice of q if $j = 0$. In other words, for $j \in I$ we have that $g \notin S(\operatorname{sgn}(f(t)\delta_j, \eta)$, where δ_j is the Dirac measure centered on j.

Define the slice $S = S(\frac{1}{|I|} \sum_{t \in I} \varepsilon_t \delta_t, \frac{\eta}{|I|})$ where $\varepsilon_t = \operatorname{sgn} f(t)$. Note that $f \in S$ and for $h \in B_X$ we have that

$$h \in S \implies h \in S(\varepsilon_t \delta_t, \eta). \tag{6.1}$$

As $g \notin S(\varepsilon_j \delta_j, \eta)$ we can conclude that $g \notin S$, by (6.1). As span $\{t^{\lambda_i}\}$ is dense, we conclude that $\sup_{h \in S} \|f - h\| < 2$ and thus f is not a delta-point.

If $|f(0)| < 1$ or $f \in M_0(\Lambda)$, the argument is similar, where the major difference is that the point q can be omitted, and the set I does not include the point 0.

$(iii) \implies (i)$. This follows directly from Proposition 6.0.6. □

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